Approximate Time-Optimal Control via Approximate Alternating Simulations

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***Abstract*— Symbolic models of control systems have recently been used to synthesize controllers enforcing specifications given by temporal logics, regular languages, or automata. These specification mechanisms can be regarded as qualitative since they divide the set of trajectories into bad trajectories (those that should be eliminated by control) and good trajectories (those that need not be eliminated). In many situations, however, a quantitative specification, where each trajectory is assigned a cost, is more appropriated. As a first step towards the synthesis of controllers enforcing qualitative and quantitative specifications we investigate in this paper the use of symbolic models for time-optimal controller synthesis. Our results show that it is possible to obtain upper and lower bounds for the time to reach a desired target by an algorithmic analysis of the symbolic model. Moreover, we can also algorithmically synthesize a feedback controller enforcing the upper bound. All the algorithms have been implemented using Binary Decision Diagrams and are illustrated by some examples.**

I. Introduction

The purpose of this paper is to advocate the use of symbolic abstractions of control systems for the synthesis of control laws enforcing, not only qualitative, but also quantitative specifications. Symbolic abstractions are simpler descriptions of control systems, typically with finitely many states, where each symbolic state represents a collection or aggregate of original states. Recent work in symbolic control [1], [2], [3] has shown that it is possible to use symbolic models to synthesize controllers enforcing specification classes that are difficult to cater using more established control theoretical methods. Examples of such specifications classes include requirements expressible in temporal logics, ω-regular languages, or automata on infinite strings. These requirements are of qualitative or binary nature since a trajectory either satisfies or does not satisfy the specification. However, in many practical situations there are reasons to prefer some trajectories over others even if all such trajectories satisfy the specification. This is typically done by associating a cost with each trajectory and thus we can regard such requirements as quantitative. As a first step towards our objective to synthesize controllers enforcing qualitative and quantitative objectives, we consider in this paper the synthesis of time-optimal controllers for reachability specifications.

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The results described in this paper are obtained by combining two different ingredients:

1) The possibility of constructing symbolic models of control systems without relying on stability assumptions as was the case in previous work [1],[2]. A thorough discussion of this result is the aim of the companion paper [4], in which similar constructions exhibiting some other interesting properties are proposed;
2) The possibility of using an alternating simulation relation from system S_a to system S_b to infer information about the solution of a time-optimal control problem on S_b from the solution of a time-optimal control problem on S_a. These results are new and reported in Section III.

The above two ingredients allow us to efficiently solve time-optimal control problems on a symbolic abstraction of a control system. In addition to synthesizing a symbolic controller providing an approximate solution for the optimal control problem, we also provide upper and lower bounds for the exact solution and show that the synthesized controller is guaranteed to enforce these bounds. A concise user guide, describing how to apply the techniques described in this paper, is provided in section IV-C.

The synthesis of optimal controllers is an old quest of the controls community and seminal contributions were made in the 60's by Pontryagin [5] and Bellman [6]. Yet, solving optimal control problems with complex specifications or complex dynamics is still a daunting problem. This motivates the interest in numerical techniques for the solution of these problems. A common method found in the literature is to directly discretize the value function and apply optimal search algorithms on graphs such as Dijstra's algorithm [7],[8]. Other techniques include Mixed (Linear or Quadratic) Integer Programing [9] and SAT-solvers [10].

The approach we follow in this paper is complementary to mentioned techniques. Instead of developing discretization techniques adapted to optimal control problems, we resort to symbolic abstractions of control systems in the spirit of [11] and analyze the simulation relations between them. Studying these relations allows us not only to provide approximate solutions to the optimal control problem, but also upper and lower quantitative bounds on the achievable performance. Moreover, through the use of the proposed abstractions many classes of dynamical systems can be accommodated, and complex qualitative specifications can be imposed. Furthermore, efficient algorithms and data structures investigated in computer science can be employed in the implementation of the proposed techniques, see for example the recent work on optimal synthesis [12]. In particular, the examples presented in the current paper were implemented using Binary Decision Diagrams [13] which can be used to automatically generate hardware [14] or software [15] implementations.

II. Preliminaries

A. Notation

Let us start by introducing some notation that will be used throughout the present paper. We denote by $\mathbb{N}$ the natural numbers including zero and by $\mathbb{N}^+$ the strictly positive natural numbers. With $\mathbb{R}^+$ we denote the strictly positive real numbers, and with $\mathbb{R}^+_0$ the positive real numbers including zero. By $\mathbb{B}$ we denote the Boolean numbers and $\mathbf{b}_n(x)$ the binary representation of x using n bits. The identity map on a set A is denoted by 1_A. If A is a subset of B we denote by $\imath_A : A \hookrightarrow B$ or simply by $\imath$ the natural inclusion map taking any $a \in A$ to $\imath(a) = a \in B$. Given a vector $x \in \mathbb{R}^n$ we denote by x_i the i–th element of x and by $\|x\|$ the infinity norm of x; we recall that $\|x\| = max\{|x_1|, |x_2|, ..., |x_n|\}$, where $|x_i|$ denotes the absolute value of x_i. The closed ball centered at $x \in \mathbb{R}^n$ with radius ε is defined by $\mathbf{B}_\varepsilon(x) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq \varepsilon\}$. We denote by $\mathbf{int}(A)$ the interior of a set A. For any $A \subseteq \mathbb{R}^n$ and $\mu \in \mathbb{R}$ we define the set $[A]_\mu = \{a \in A \mid a_i = k_i\mu, k_i \in \mathbb{Z}, i = 1, ..., n\}$. The set $[A]_\mu$ will be used as an approximation of the set A with precision μ. Geometrically, for any $\mu \in \mathbb{R}^+$ and $\lambda \geq \mu/2$ the collection of sets $\{\mathbf{B}_\lambda(q)\}_{q\in[\mathbb{R}^n]_\mu}$ is a covering of $\mathbb{R}^n$. A continuous function $\gamma : \mathbb{R}^+_0 \to \mathbb{R}^+_0$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. We identify a relation $R \subseteq A \times B$ with the map $R : A \to 2^B$ defined by $b \in R(a)$ iff $(a, b) \in R$. Also, R^{-1} denotes the inverse relation defined by $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$. We also denote by $\mathbf{d} : X \times X \to \mathbb{R}^+_0$ a metric in the space X.

B. Systems

In the present paper we use the mathematical abstraction of *systems* to model dynamical phenomena. This abstraction is formalized in the following definition:

Definition II.1 (System [11]). *A system S is a sextuple: $(X, X_0, U, \longrightarrow, Y, H)$ consisting of:*

- *a set of states X;*
- *a set of initial states $X_0 \subseteq X$*
- *a set of inputs U;*
- *a transition relation $\longrightarrow \subseteq X \times U \times X$;*
- *a set of outputs Y;*
- *an output map $H : X \to Y$.*

A system $(X, X_0, U, \longrightarrow, Y, H)$ is said to be:

- *metric, if the output set Y is equipped with a metric $\mathbf{d} : Y \times Y \to \mathbb{R}^+_0$;*
- *countable, if X and U are countable sets;*
- *finite, if X and U are finite sets.*

We will often use the notation $x \xrightarrow{u} y$ to denote $(x, u, y) \in \longrightarrow$. For a transition $x \xrightarrow{u} y$, state y is called a u-successor, or simply successor. We denote the set of u-successors of a state x by $\mathrm{Post}_u(x)$. If for all initial states x and inputs u the sets $\mathrm{Post}_u(x)$ are singletons (or empty sets) we will say the system S is *deterministic*, if on the other hand for some state x and input u the set $\mathrm{Post}_u(x)$ has cardinality greater than one, we will say that system S is *non-deterministic*. Furthermore, if there exists some pair (x, u) such that $\mathrm{Post}_u(x) = \emptyset$ we say the system is *blocking*, and otherwise *non-blocking*. We also use the notation $U(x)$ to denote the set $U(x) = \{u \in U | \mathrm{Post}_u(x) \neq \emptyset\}$.

We can also define a deterministic version of system S_a, which we will denote $S_{d(a)}$ by extending the set of inputs:

Definition II.2. *The deterministic system:*

$$S_{d(a)} = (X_a, X_{a0}, U_{d(a)}, \underset{d(a)}{\longrightarrow}, Y_a, H_a)$$

associated to a system $S_a = (X_a, X_{a0}, U_a, \underset{a}{\longrightarrow}, Y_a, H_a)$, is defined by:

- $U_{d(a)} = U_a \times X_a$
- $x \xrightarrow[d(a)]{(v,x')} x'$ *if there exists* $x \xrightarrow[a]{v} x'$

Sometimes we need to refer to the possible sequences of states and/or outputs that a system can exhibit. We call these sequences of states or outputs: *behaviours*. Formally, behaviours are defined as follows:

Definition II.3 (Behaviours [11]). *For a system S and given any state $x \in X$, a* finite internal behaviour *generated from x is a finite sequence of transitions:*

$$x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} x_2 \xrightarrow{u_2} \ldots \xrightarrow{u_{n-2}} x_{n-1} \xrightarrow{u_{n-1}} x_n$$

such that $x_0 = x$ and $x_{i-1} \xrightarrow{u_{i-1}} x_i$ for all $0 \leq i < n$. Through the output map, every finite internal behaviour defines a finite external behaviour*:*

$$y_0 \longrightarrow y_1 \longrightarrow y_2 \longrightarrow \ldots \longrightarrow y_{n-1} \longrightarrow y_n$$

with $H(x_i) = y_i$ for all $0 \leq i < n$.

An infinite internal behaviour *generated from x is an infinite sequence of transitions:*

$$x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} x_2 \xrightarrow{u_2} x_3 \xrightarrow{u_3} \ldots$$

such that $x_0 = x$ and $x_{i-1} \xrightarrow{u_{i-1}} x_i$ for all $i \in \mathbb{N}$. Through the output map, every infinite internal behaviour defines an infinite external behaviour*:*

$$y_0 \longrightarrow y_1 \longrightarrow y_2 \longrightarrow y_3 \longrightarrow \ldots$$

with $H(x_i) = y_i$ for all $i \in \mathbb{N}$.

By $\mathcal{B}_x(S)$ ($\mathcal{B}^\omega_x(S)$), we denote the set of finite (infinite) external behaviours generated from x. Sometimes we use the notation $\mathbf{y} = y_0y_1y_2 \ldots y_n$, to denote external behaviours. A behaviour $\mathbf{y}$ is said to be *maximal* if there is no other behaviour containing $\mathbf{y}$ as a prefix.

In this paper we consider control systems to describe dynamics evolving continuously on time over an infinite set of states (*e.g.* $\mathbb{R}^n$). Control systems are formalized in the following definition:

Definition II.4 (Continuous-time control system [11])**.** *A control system is a triple* $\Sigma = (\mathbb{R}^n, \mathcal{U}, f)$ *consisting of:*

- *the state set* $\mathbb{R}^n$*;*
- *a set of input curves* $\mathcal{U}$ *whose elements are essentially bounded piece-wise continuous functions of time from intervals of the form* $]a, b[\subseteq \mathbb{R}$ *to* $U \subseteq \mathbb{R}^m$ *with* $a < 0 < b$*;*
- *a smooth map* $f : \mathbb{R}^n \times U \to \mathbb{R}^n$.

A piecewise continuously differentiable curve $\xi :]a, b[\to \mathbb{R}^n$ *is said to be a trajectory or solution of* Σ *if there exists* $\upsilon \in \mathcal{U}$ *satisfying:*

$$\dot{\xi}(t) = f(\xi(t), \upsilon(t)),$$

for almost all $t \in]a, b[$. *Control system* Σ *is said to be forward complete if every trajectory is defined on an interval of the form* $]a, \infty[$.

Although we have defined trajectories over open domains, we shall refer to trajectories $\xi : [0, \tau] \to \mathbb{R}^n$ defined on closed domains $[0, \tau]$, $\tau \in \mathbb{R}^+$ with the understanding of the existence of a trajectory $\xi' :]a, b[\to \mathbb{R}^n$ such that $\xi = \xi'|_{[0,\tau]}$. We will also write $\xi_{x\upsilon}(t)$ to denote the point reached at time $t \in [0, \tau]$ under the input υ from initial condition x; this point is uniquely determined, since the assumptions on f ensure existence and uniqueness of trajectories.

C. Systems relations

In the following sections we introduce abstractions for control systems. The results we prove build upon certain relations that can be established between these models. These relations are formalized through the following two definitions:

Definition II.5 (Approximate Simulation Relation [11])**.** *Consider two metric systems* S_a *and* S_b *with* $Y_a = Y_b$*, and let* $\varepsilon \in \mathbb{R}_0^+$. *A relation* $R \subseteq X_a \times X_b$ *is an* ε-approximate simulation relation *from* S_a *to* S_b *if the following three conditions are satisfied:*

1) *for every* $x_{a0} \in X_{a0}$*, there exists* $x_{b0} \in X_{b0}$ *with* $(x_{a0}, x_{b0}) \in R$*;*
2) *for every* $(x_a, x_b) \in R$ *we have* $\mathbf{d}(H_a(x_a), H_b(x_b)) \leq \varepsilon$*;*
3) *for every* $(x_a, x_b) \in R$ *we have that:* $x_a \xrightarrow[a]{u_a} x'_a$ *in* S_a *implies the existence of* $x_b \xrightarrow[b]{u_b} x'_b$ *in* S_b *satisfying* $(x'_a, x'_b) \in R$.

We say that S_a *is* ε*-approximately simulated by* S_b *or that* S_b ε*-approximately simulates* S_a*, denoted by* $S_a \preceq_{\mathcal{S}}^{\varepsilon} S_b$*, if there exists an* ε*-approximate simulation relation from* S_a *to* S_b.

Definition II.6 (Approximate alternating simulation relation [11])**.** *Let* S_a *and* S_b *be metric systems with* $Y_a = Y_b$ *and let* $\varepsilon \in \mathbb{R}_0^+$. *A relation* $R \subseteq X_a \times X_b$ *is an* ε-approximate alternating simulation relation *from* S_a *to* S_b *if the following three conditions are satisfied:*

1) *for every* $x_{a0} \in X_{a0}$ *there exists* $x_{b0} \in X_{b0}$ *with* $(x_{a0}, x_{b0}) \in R$*;*
2) *for every* $(x_a, x_b) \in R$ *we have* $\mathbf{d}(H_a(x_a), H_b(x_b)) \leq \varepsilon$*;*
3) *for every* $(x_a, x_b) \in R$ *and for every* $u_a \in U_a(x_a)$ *there exists* $u_b \in U_b(x_b)$ *such that for every* $x'_b \in Post_{u_b}(x_b)$ *there exists* $x'_a \in Post_{u_a}(x_a)$ *satisfying* $(x'_a, x'_b) \in R$.

We say that S_a *is* ε*-approximately alternatingly simulated by* S_b *or that* S_b ε*-approximately alternatingly simulates* S_a*, denoted by* $S_a \preceq_{\mathcal{AS}}^{\varepsilon} S_b$*, if there exists an* ε*-approximate alternating simulation relation from* S_a *to* S_b.

Note that whenever systems are deterministic the notion of alternating simulation degenerates into that of simulation. Also note that for any system S_a, its deterministic counterpart $S_{d(a)}$ satisfies $S_a \preceq_{\mathcal{AS}}^{0} S_{d(a)}$.

III. Time-optimal control

A. Problem definition

In the present section we introduce general time-optimal control problems over general systems, which are the objects of our study. Before formalizing this problem we need to introduce some more notation.

For general systems, the intuitive notion of *feedback composition* of a system S with another system S_c is denoted by $S_c \times_{\mathcal{F}} S$. The reader can find a formal definition of feedback composition and a study of its properties in [11]. We shall omit the formal definition in this paper for space reasons and since we will not need it in any technical argument. Feedback composition is now used to define reachability problems:

Problem III.1 (Reachability)**.** *Let* S_a *be a system with* $Y_a = X_a$ *and* $H_a = 1_{X_a}$*, and let* $W \subseteq X_a$ *be a set of states. The reachability problem asks to find a controller* S_c *such that:*

- S_c *is feedback composable with* S_a*;*
- *for every maximal behaviour* $\mathbf{y} \in \mathcal{B}_{x_0}(S_c \times_{\mathcal{F}} S_a) \cup \mathcal{B}^{\omega}_{x_0}(S_c \times_{\mathcal{F}} S_a)$ *there exists* $k(x_0) \in \mathbb{N}$ *such that* $\mathbf{y}(k(x_0)) = y_{k(x_0)} \in W$*;*

To simplify the presentation, we consider only systems in which $X_a = Y_a$ and $H_a = 1_{X_a}$. However, all the results in this paper can be extended to systems with $X_a \neq Y_a$ and $H_a \neq 1_{X_a}$ by using the techniques described in [11]. We denote by $\mathcal{R}(S_a, W)$ the set of controllers that solve the reachability problem for system S_a with the target set W as specification.

Definition III.2 (Entry time)**.** *The* entry time *of* $S_c \times_{\mathcal{F}} S_a$ *into* W *from* x_0*, denoted by* $J(S_c \times_{\mathcal{F}} S_a, W, x_0)$*, is the minimum* $k \in \mathbb{N}$ *such that* $\forall\, \mathbf{y} \in \mathcal{B}_{x_0}(S_c \times_{\mathcal{F}} S_a) \cup \mathcal{B}^{\omega}_{x_0}(S_c \times_{\mathcal{F}} S_a)$*, there exists some* $k' \in [0, k]$ *such that* $\mathbf{y}(k') = y_{k'} \in W$.

If the set W is not reachable from state x_0 using controller S_c we define $J(S_c \times_{\mathcal{F}} S_a, W, x_0) = \infty$. Note that asking in the definition for the *minimum* k is needed because in general $S_c \times_{\mathcal{F}} S_a$ might be a non-deterministic system, and thus there might be more than one behaviour contained in $\mathcal{B}_{x_0}(S_c \times_{\mathcal{F}} S_a) \cup \mathcal{B}^{\omega}_{x_0}(S_c \times_{\mathcal{F}} S_a)$.

Now we can formulate the time-optimal control problem in terms of systems as follows:

Problem III.3 (Time-optimal control)**.** *Let S_a be a system with $Y_a = X_a$ and $H_a = 1_{X_a}$, and let $W \subseteq X_a$ be a subset of the set of states of S_a. Find the controller $S_c^* \in \mathcal{R}(S_a, W)$ such that for any other controller $S_c \in \mathcal{R}(S_a, W)$ the following is satisfied:*

$$\forall x_0 \in X_{a0}, \quad J(S_c \times_{\mathcal{F}} S_a, W, x_0) \geq J(S_c^* \times_{\mathcal{F}} S_a, W, x_0).$$

B. Cost bounds

The entry time J acts as the cost function we aim at minimizing by designing an appropriate controller. We establish now a result that will help us later in providing bounds on the achievable cost.

Theorem III.4. *Let S_a and S_b be two metric systems with $Y_a = Y_b$ and the same metric. If the following conditions are satisfied:*

- *there exists a relation $R_\varepsilon \subseteq X_a \times X_b$ such that $S_a \preceq^{\varepsilon}_{\mathcal{AS}} S_b$;*
- *$(x_{a0}, x_{b0}) \in R_\varepsilon$;*
- *and for all $x_a \in W_a$ there exists $x_b \in W_b$ such that $(x_a, x_b) \in R_\varepsilon$*

then the following holds:

$$J(S_{cb}^* \times_{\mathcal{F}} S_b, W_b, x_{b0}) \leq J(S_{ca}^* \times_{\mathcal{F}} S_a, W_a, x_{a0})$$

where $S_{ca}^* \in \mathcal{R}(S_a, W_a)$ and $S_{cb}^* \in \mathcal{R}(S_b, W_b)$ denote the optimal controllers for their respective time-optimal control problems.

Proof: We proceed by contradiction. Assume $J(S_{ca}^* \times_{\mathcal{F}} S_a, W_a, x_{a0}) < J(S_{cb}^* \times_{\mathcal{F}} S_b, W_b, x_{b0})$. From $S_a \preceq^{\varepsilon}_{\mathcal{AS}} S_b$ we have (see Proposition 11.10 in [11] and discussion thereafter) that the system $S_c' = S_{ca}^* \times_{\mathcal{F}} S_a$ is a controller for S_b and $S_c' \times^{\varepsilon}_{\mathcal{G}} S_b \preceq^{\frac{1}{2}\varepsilon}_{\mathcal{S}} S_{ca}^* \times_{\mathcal{F}} S_a = S_c'$. But then, from the third assumption, we have that for all $x_{a0} \in X_{a0}$ and $x_{b0} \in X_{b0}$ such that $(x_{a0}, x_{b0}) \in R_\varepsilon$, the following holds: $J(S_c' \times_{\mathcal{F}} S_b, W_b, x_{b0}) \leq J(S_{ca}^* \times_{\mathcal{F}} S_a, W_a, x_{a0})$. Hence, contradicting that $S_{cb}^* \in \mathcal{R}(S_b, W_b)$ is an optimal controller for the reachability problem with system S_b and target set W_b. □

C. Solution to the optimal control problem

We show now that there exists a fixed point algorithm solving the reachability problem. Moreover, the solutions obtained in this way are, by construction, optimal controllers for the time-optimal reachability problem.

For a given system S_a and target set $W \subseteq X_a$, we define the operator $G_W : 2^{X_a} \to 2^{X_a}$ by:

$$G_W(Z) = \{x_a \in X_a \mid x_a \in W \text{ or } \exists\, u_a \in U_a(x_a) \text{ s.t. } \emptyset \neq \mathrm{Post}_{u_a}(x_a) \subseteq Z\}$$

An optimal controller for system S_a to reach the set W exists if and only if the minimal fixed point $Z = \lim_{i \to \infty} G_W^i(\emptyset)$ satisfies $Z \cap X_{a0} \neq \emptyset$. Using the operator G_W again the optimal controller $S_c^* \in \mathcal{R}(S_a, W)$:

$$S_c^* = (X_c, X_{c0}, U_a, \underset{c}{\longrightarrow}, X_c, 1_{X_c})$$

is defined as:

- $X_c = Z_c$;
- $X_{c0} = Z \cap X_{a0}$;
- $x_c \xrightarrow[c]{u_a} x_c'$ if there exists a $k \in \mathbb{N}^+$ such that $x_c \notin G_W^k(\emptyset)$ and $\emptyset \neq \mathrm{Post}_{u_a}(x_c) \subseteq G_W^k(\emptyset)$

where $\mathrm{Post}_{u_a}(x_c)$ refers to the u_a–successors in S_a.

For more details about this controller design we refer the reader to Chapter 6 of [11].

IV. Approximate time-optimal control

A. Symbolic models for control

In the subsequent sections we will assume that the control systems under consideration satisfy the following assumption:

Assumption IV.1. *The control system Σ is incrementally forward complete, i.e. there exists a continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each $t \geq 0$, such that for any two initial conditions $x_1, x_2 \in X_0$, and for any $\tau \in \mathbb{R}_0^+$ the following bound holds:*

$$\|\xi_{x_1 \upsilon}(\tau) - \xi_{x_2 \upsilon}(\tau))\| \leq \beta(\|x_1 - x_2\|, \tau).$$

Our goal is to provide solutions to time-optimal control problems in an automatic fashion by means of computational tools. In order to obtain finite models to which we can apply computational algorithms we start by defining models for control systems that evolve in discrete time:

Definition IV.2. *The system*

$$S_\tau = (X_\tau, X_{\tau 0}, U_\tau, \underset{\tau}{\longrightarrow}, Y_\tau, H_\tau)$$

associated with a control system $\Sigma = (\mathbb{R}^n, \mathcal{U}, f)$ and with $\tau \in \mathbb{R}^+$ consists of:

- *$X_\tau = \mathbb{R}^n$;*
- *$X_{\tau 0} = X_\tau$;*
- *$U_\tau = \{\upsilon \in \mathcal{U} \mid \mathrm{dom}\, \upsilon = [0, \tau]\}$;*
- *$x \xrightarrow[\tau]{\upsilon} x'$ if there exist $\upsilon \in U_\tau$, and a trajectory $\xi_{x\upsilon} : [0, \tau] \to \mathbb{R}^n$ of Σ satisfying $\xi_{x\upsilon}(\tau) = x'$;*
- *$Y_\tau = \mathbb{R}^n$;*
- *$H_\tau = 1_{\mathbb{R}^n}$.*

The output set $Y_\tau = \mathbb{R}^n$ of $S_\tau(\Sigma)$ is naturally equipped with the norm-induced metric $\mathbf{d}(y, y') = \|y - y'\|$.

Note how the models introduced above are still infinite (they have an infinite state set). We now further quantize $S_\tau(\Sigma)$ to construct a system $S_{\tau\eta}(\Sigma)$ with a countable state set. Moreover, we assume that the same input sets are available for $S_\tau(\Sigma)$ and its quantized counterpart $S_{\tau\eta}(\Sigma)$. This assumption is made for clarity of exposition, while it also models realistic scenarios in which the controller only admits (a finite number of) digital inputs, i.e. piecewise constant and quantized. Yet, all the above theorems can be modified to accommodate different input sets, as long as the set of inputs available for the symbolic abstraction $S_{\tau\eta}(\Sigma)$ is "rich enough" to approximate the original input set. Moreover, all the results that follow in subsequent sections are independent of this assumption and are solely based on the relations we prove in this subsection.

Definition IV.3. *The system*

$$S_{\tau\eta} = (X_{\tau\eta}, X_{\tau\eta 0}, U_{\tau\eta}, \underset{\tau\eta}{\longrightarrow}, Y_{\tau\eta}, H_{\tau\eta})$$

associated with a control system $\Sigma = (\mathbb{R}^n, \mathcal{U}, f)$ and with $\tau, \eta \in \mathbb{R}^+$ consists of:

- $X_{\tau\eta} = [\mathbb{R}^n]_\eta$;
- $X_{\tau\eta 0} = X_{\tau\eta}$
- $U_{\tau\eta} = \{\upsilon \in \mathcal{C} \mid \operatorname{dom} \upsilon = [0, \tau]\}$;
- $x \xrightarrow[\tau\eta]{\upsilon} x'$ *if there exist* $\upsilon \in U_{\tau\eta}$, *and a trajectory* $\xi_{x\upsilon} : [0, \tau] \to \mathbb{R}^n$ *of* Σ *satisfying* $\mathbf{int}(\mathbf{B}_{\beta(\eta/2,\tau)}(\xi_{x\upsilon}(\tau)) \cap \mathbf{B}_{\eta/2}(x')) \neq \emptyset$;
- $Y_{\tau\eta} = \mathbb{R}^n$;
- $H_{\tau\eta} = \imath : X_{\tau\eta} \hookrightarrow \mathbb{R}^n$.

The system $S_{\tau\eta}(\Sigma)$ can be regarded as a time and space quantization of a control system Σ. It is constructed by approximating the transitions of $S_\tau(\Sigma)$ so as to enforce departure from and arrival at states in $X_{\tau\eta} = [\mathbb{R}^n]_\eta$. The domain of evolution of this abstraction is only countable but infinite in general. In order to obtain abstractions resulting in finite systems, one approach is to restrict the domain $X_{\tau\eta}$ to a finite subset of $[\mathbb{R}^n]_\eta$. In many practical applications there are indeed physical or technological limitations imposing boundaries on the state set. Note also that $S_{\tau\eta}(\Sigma)$ is, in general, a nondeterministic system.

In the following theorem we establish relationships between the systems $S_\tau(\Sigma)$ and its quantized counterpart $S_{\tau\eta}(\Sigma)$:

Theorem IV.4. *For any control system Σ satisfying Assumption IV.1, given a desired precision $\varepsilon \in \mathbb{R}^+$, for any $\tau \in \mathbb{R}^+$, and for $\eta = 2\varepsilon$, the following holds:*

$$S_{\tau\eta}(\Sigma) \preceq^{\varepsilon}_{\mathcal{AS}} S_\tau(\Sigma) \preceq^{\varepsilon}_{\mathcal{S}} S_{\tau\eta}(\Sigma).$$

Proof: **($S_{\tau\eta}(\Sigma) \preceq^{\varepsilon}_{\mathcal{AS}} S_\tau(\Sigma)$):** Consider the relation $R_\eta \subseteq X_\tau \times X_{\tau\eta}$ defined by $(x_\tau, x_{\tau\eta}) \in R_\varepsilon$ if and only if $\|x_\tau - x_{\tau\eta}\| \leq \varepsilon$. Conditions 1. and 2. in Definition II.6 are automatically satisfied from the definition of R_ε, and $X_{\tau\eta 0} = [X_{\tau 0}]_\eta$. To prove that the third condition is satisfied consider a pair $(x_\tau, x_{\tau\eta}) \in R_\varepsilon$. For any $\upsilon_{\tau\eta} \in U_{\tau\eta}$ there exists $\upsilon_\tau \in U_\tau$ such that $\upsilon_\tau = \upsilon_{\tau\eta}$. The element $x'_\tau = \xi_{x_\tau \upsilon_\tau}(\tau)$ is the only element in $\mathrm{Post}_{\upsilon_\tau}(x_\tau)$, and, in virtue of Assumption IV.1 and $\eta = 2\varepsilon$, it also satisfies $x'_\tau \in \mathbf{B}_{\beta(\eta/2,\tau)}(\xi_{x_{\tau\eta}\upsilon_{\tau\eta}}(\tau))$. Hence, by construction of $S_{\tau\eta}$ there exists $x'_{\tau\eta} \in \mathrm{Post}_{\upsilon_{\tau\eta}}(x_{\tau\eta})$ such that $\|x'_{\tau\eta} - x'_\tau\| \leq \eta/2 = \varepsilon$, i.e. $(x'_\tau, x'_{\tau\eta}) \in R_\varepsilon$.

($S_\tau(\Sigma) \preceq^{\varepsilon}_{\mathcal{S}} S_{\tau\eta}(\Sigma)$): Consider the same relation R_ε as in the first part of the proof. We now show that R_ε is an ε–approximate simulation relation from $S_\tau(\Sigma)$ to $S_{\tau\eta}(\Sigma)$. Conditions 1. and 2. in Definition II.5 are again trivially satisfied. To show that condition 3. in Definition II.5 also holds consider any $(x_\tau, x_{\tau\eta}) \in R_\varepsilon$, $\upsilon_\tau \in U_\tau$ and the transition $x_\tau \xrightarrow[\tau]{\upsilon_\tau} x'_\tau$ in $S_\tau(\Sigma)$. Since $U_{\tau\eta} = U_\tau$ there exists $\upsilon_{\tau\eta} \in U_{\tau\eta}$, $\upsilon_{\tau\eta} = \upsilon_\tau$, and thus, in virtue of Assumption IV.1 and $\eta = 2\varepsilon$, it follows that $x'_\tau \in \mathbf{B}_{\beta(\eta/2,\tau)}(\xi_{x_{\tau\eta}\upsilon_{\tau\eta}}(\tau))$. By construction there exists $x'_{\tau\eta} \in \mathrm{Post}_{\upsilon_{\tau\eta}}(x_{\tau\eta})$ such that $\|x'_{\tau\eta} - x'_\tau\| \leq \eta/2 = \varepsilon$, hence $(x'_\tau, x'_{\tau\eta}) \in R_\varepsilon$. □

Similar results can be obtained with constructions of $S_{\tau\eta}$ slightly different as presented in more detail in the companion paper [4].

Following the definition of alternating simulation relation, we obtain as a trivial consequence of the preceding result the following corollary:

Corollary IV.5. *For any control system Σ satisfying Assumption IV.1, given a desired precision $\varepsilon \in \mathbb{R}^+$, for any $\tau \in \mathbb{R}^+$, and for $\eta = 2\varepsilon$, the following holds:*

$$S_\tau(\Sigma) \preceq^{\varepsilon}_{\mathcal{AS}} S_{d(\tau\eta)}(\Sigma).$$

In the results hereafter we use the following extra assumption:

Assumption IV.6. *The function β in Assumption IV.1 is* superlinear *on its first argument, i.e.*

$$\beta(a, t) + \beta(b, t) \leq \beta(a + b, t)$$

for each $t \geq 0$.

We will comment on this assumption after the following theorem describing how to relate abstractions obtained for different values of η:

Theorem IV.7. *For any control system Σ satisfying Assumption IV.1, any $\eta \in \mathbb{R}^+$, any $\tau \in \mathbb{R}^+$, and any $\eta' = \frac{\eta}{\rho}$ with ρ an odd number greater than one, the following holds:*

$$S_{\tau\eta}(\Sigma) \preceq^{\varepsilon}_{\mathcal{AS}} S_{\tau\eta'}(\Sigma)$$

with $\varepsilon = \frac{\eta - \eta'}{2}$.

Proof: Consider the relation $R_\varepsilon \subseteq X_{\tau\eta} \times X_{\tau\eta'}$ defined by $(x_{\tau\eta}, x_{\tau\eta'}) \in R_\varepsilon$ iff $\|x_{\tau\eta} - x_{\tau\eta'}\| \leq \varepsilon$. We now show that R_ε is an ε–approximate alternating simulation relation from $S_{\tau\eta}(\Sigma)$ to $S_{\tau\eta'}(\Sigma)$. Conditions 1. and 2. in Definition II.6 are automatically satisfied by the definition of R_ε. In order to show that condition 3. is also satisfied, let $(x_{\tau\eta}, x_{\tau\eta'}) \in R_\varepsilon$ and pick any $\upsilon_{\tau\eta} \in U_{\tau\eta}$. The spaces of inputs are the same in both systems, therefore there exists $\upsilon_{\tau\eta'} = \upsilon_{\tau\eta}$, $\upsilon_{\tau\eta'} \in U_{\tau\eta'}$. By construction every $x'_{\tau\eta'} \in \mathrm{Post}_{\upsilon_{\tau\eta'}}(x_{\tau\eta'})$ satisfies $\mathbf{int}(\mathbf{B}_{\eta'/2}(x'_{\tau\eta'}) \cap \mathbf{B}_{\beta(\eta'/2,\tau)}(\xi_{x_{\tau\eta'}\upsilon_{\tau\eta'}}(\tau))) \neq \emptyset$. Furthermore, from Assumption IV.1 we have that $\|\xi_{x_{\tau\eta'}\upsilon_{\tau\eta'}}(\tau) - \xi_{x_{\tau\eta'}\upsilon_{\tau\eta}}(\tau)\| \leq \beta(\varepsilon, \tau)$, and from Assumption IV.6 we also know that $\beta(\eta'/2, \tau) + \beta(\varepsilon, \tau) \leq \beta(\eta/2, \tau)$. Thus $\mathbf{B}_{\beta(\eta'/2,\tau)}(\xi_{x_{\tau\eta'}\upsilon_{\tau\eta'}}(\tau)) \subset \mathbf{B}_{\beta(\eta/2,\tau)}(\xi_{x_{\tau\eta}\upsilon_{\tau\eta}}(\tau))$. Now notice that, for ρ odd, $[X]_{\eta'}$ defines a sub-grid of $[X]_\eta$ in which if for any $x'_k \in [X]_{\eta'}$, $x_k \in [X]_\eta$, the following holds: if $\mathbf{int}(\mathbf{B}_{\eta'/2}(x'_k) \cap \mathbf{B}_{\eta/2}(x_k)) \neq \emptyset$, then there is no other $x_l \in [X]_\eta$, with $x_l \neq x_k$ such that $\mathbf{int}(\mathbf{B}_{\eta'/2}(x'_k) \cap \mathbf{B}_{\eta/2}(x_l)) \neq \emptyset$. Moreover any pair $x_k \in [X]_\eta$, $x'_k \in [X]_{\eta'}$ satisfying $\mathbf{int}(\mathbf{B}_{\eta'/2}(x'_k) \cap \mathbf{B}_{\eta/2}(x_k)) \neq \emptyset$, also satisfy $\|x_k - x'_k\| \leq \varepsilon$. Thus for every $x'_{\tau\eta'} \in \mathrm{Post}_{\upsilon_{\tau\eta'}}(x_{\tau\eta'})$ there exists $z_{\tau\eta} \in [X]_{\tau\eta}$ such that $\mathbf{int}(\mathbf{B}_{\eta'/2}(x'_{\tau\eta'}) \cap \mathbf{B}_{\eta/2}(z_{\tau\eta})) \neq \emptyset$, and $\|x'_{\tau\eta'} - z_{\tau\eta}\| \leq \varepsilon \leq \eta/2$. But then $\mathbf{int}(\mathbf{B}_{\beta\eta/2,\tau}(z_{\tau\eta}) \cap \mathbf{B}_{\beta(\eta/2,\tau)}(\xi_{x_{\tau\eta}\upsilon_{\tau\eta}}(\tau))) \neq \emptyset$, and thus $z_{\tau\eta} \in \mathrm{Post}_{\upsilon_{\tau\eta}}(x_{\tau\eta})$, which implies the existence of $x'_{\tau\eta} \in \mathrm{Post}_{\upsilon_{\tau\eta}}(x_{\tau\eta})$ such that $(x'_{\tau\eta}, x'_{\tau\eta'}) \in R_\varepsilon$, namely $x'_{\tau\eta} = z_{\tau\eta}$. □

Combining this result with the fact that $S_{\tau\eta'}(\Sigma) \preceq^0_{\mathcal{AS}} S_{d(\tau\eta')}(\Sigma)$ we obtain trivially the following result:

Corollary IV.8. *For any control system Σ satisfying Assumption IV.1, any $\eta \in \mathbb{R}^+$, any $\tau \in \mathbb{R}^+$, and any $\eta' = \frac{\eta}{\rho}$ with ρ an odd number greater than one, the following holds:*

$$S_{\tau\eta}(\Sigma) \preceq^\varepsilon_{\mathcal{AS}} S_{d(\tau\eta')}(\Sigma)$$

with $\varepsilon = \frac{\eta - \eta'}{2}$.

The following remark establishes that Assumption IV.6 is not as restrictive as it might look at a first glance:

Remark IV.9. *Given a desired precision ε, Theorem IV.4 establishes a maximum value for η: $\eta \leq 2\varepsilon$. If for a fixed τ a function $\beta(\cdot,\tau) : \mathbb{R}^+_0 \to \mathbb{R}^+_0$, $\beta(\cdot,\tau) \in \mathcal{K}_\infty$ is available, on the closed positive interval $[0,\eta]$ it is always possible to find a superlinear function $\beta_q(\cdot,\tau) : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ such that $\beta(s,\tau) \leq \beta_q(s,\tau)$ for all $s \in [0,\eta]$. As a trivial example, if $\beta(\cdot,\tau)$ is differentiable consider the following linear function $\beta_q(\cdot,\tau) = s(\max_{t\in[0,\eta]} \frac{d}{dt}\beta(t,\tau))$ bounding $\beta(\cdot,\tau)$.*

B. Approximate time-optimal control via symbolic models

In the previous section we concluded that the system $S_{\tau\eta}(\Sigma)$ is such that $S_{\tau\eta}(\Sigma) \preceq^\eta_{\mathcal{AS}} S_\tau(\Sigma) \preceq^\eta_{\mathcal{AS}} S_{d(\tau\eta)}(\Sigma)$. We would like to solve a time-optimal control problem over $S_\tau(\Sigma)$ by resorting to the approximate model $S_{\tau\eta}(\Sigma)$ in which computational tools can be employed. Moreover, we would like to obtain bounds for the true optimal cost in order to assess the quality of the solutions obtained after refining the controllers obtained over $S_{\tau\eta}(\Sigma)$ to $S_\tau(\Sigma)$.

In what follows we require the following definitions concerning approximations of sets:

Definition IV.10 (η-Inner (Outer) approximations of sets)**.** *The sets $\lfloor W \rfloor_\eta$,$\lceil W \rceil_\eta$ are defined as the η-Inner (Outer) approximations of a given set $W \subseteq X \subseteq \mathbb{R}^n$ as formalized by:*

$$\begin{aligned} \lfloor W \rfloor_\eta &= \{x \in [X]_\eta | \mathbf{B}_{\eta/2}(x) \subseteq W\}, \\ \lceil W \rceil_\eta &= \{x \in [X]_\eta | \mathbf{B}_{\eta/2}(x) \cap W \neq \emptyset\}. \end{aligned}$$

Note that if now we define the relation $R_\eta \subset X \times [X]_\eta$, $X \subseteq \mathbb{R}^n$ as $(x, x_\eta) \in R_\eta \Leftrightarrow \|x - x_\eta\| \leq \eta/2$, we have $R_\eta^{-1}(\lfloor W \rfloor_\eta) \subseteq W$ and $R_\eta(W) \subseteq \lceil W \rceil_\eta$.

With all these definitions in place we are ready to establish one of the main results of the present work:

Theorem IV.11. *Consider a control system Σ satisfying Assumption IV.1, if $\|x_{\tau 0} - x_{\tau\eta 0}\| \leq \eta/2$ the following bounds hold:*

$$\begin{aligned} J(S^*_c \times_\mathcal{F} S_{\tau\eta}(\Sigma), \lfloor W \rfloor_\eta, x_{\tau\eta 0}) &\geq J(S^*_{c\tau} \times_\mathcal{F} S_\tau(\Sigma), W, x_{\tau 0}) \\ J(S^*_{cd(\tau\eta)} \times_\mathcal{F} S_{d(\tau\eta)}(\Sigma), \lceil W \rceil_\eta, x_{\tau\eta 0}) &\leq J(S^*_{c\tau} \times_\mathcal{F} S_\tau(\Sigma), W, x_{\tau 0}) \end{aligned}$$

where $S^*_{c\tau} \in \mathcal{R}(S_\tau(\Sigma), W)$, $S^*_{c\tau\eta} \in \mathcal{R}(S_{\tau\eta}(\Sigma), \lfloor W \rfloor_\eta)$ and $S^*_{cd(\tau\eta)} \in \mathcal{R}(S_{d(\tau\eta)}(\Sigma), \lceil W \rceil_\eta)$ are the optimal controllers for their respective time-optimal control problems.

Proof: This theorem is a direct consequence of applying Theorem IV.4, Corollary IV.5 and Theorem III.4. □

C. Approximate time-optimal control in practice

In this section we present a typical sequence of steps to be followed when applying the presented techniques in practice.

1) **Select a desired precision ε.** This precision is in general given by specified practical margins of error.
2) **Enlarge the target set W.** Enlarge the target set according to the desired error margins.
3) **Compute $J(S^*_{cd(\tau\eta)} \times_\mathcal{F} S_{d(\tau\eta)}(\Sigma), \lceil W \rceil_\eta, x_{\eta 0})$ (lower bound on the cost).** This bound is obtained through the use of the fixed-point algorithm in Section III-C. This is the best lower bound one can obtain since it follows from Theorem III.4 that by reducing η we will not obtain a better lower bound.
4) **Compute $J(S^*_c \times_\mathcal{F} S_{\tau\eta}(\Sigma), \lfloor W \rfloor_\eta, x_{\eta 0})$ (upper bound on the cost).** This bound is computed using the fixed-point algorithm in Section III-C. The controller obtained when computing this bound, *i.e.* S^*_c, is the optimal controller for $S_{\tau\eta}(\Sigma)$ and approximately optimal for $S_\tau(\Sigma)$.
5) **Iterate.** If the obtained upper bound is not acceptable, reduce η according to $\eta' = \frac{\eta}{\rho}$ with an odd $\rho > 1$, and recompute the controller and upper bound. In virtue of Theorems IV.7 and III.4, by reducing η the upper bound will not increase. Moreover, it is our experience that, in general, the upper bound will be reduced by reducing η.

V. Implementation and example

A. Binary Decisions Diagrams

Binary Decision Diagrams (BDDs) are efficient data structures used to store boolean functions. Intuitively, a BDD is a binary tree with as many levels as bits in the domain of the boolean function δ_a to be represented. The tree has two final leaves labeled *true* and *false*, representing the output of δ_a. At level i a branch is selected depending on the value of the i-th bit of the input to δ_a until a final leave is reached. BDD representations exhibit many advantages for verification purposes [13]. We remark their effective use of space when using their canonical form: *Reduced Ordered BDD (ROBDD)* [13].

We employ BDDs to represent finite systems by transforming the transition relation into a boolean function. If for a given system S_a we know that the cardinalities of X_a and U_a are $|X_a| \leq 2^{n_x}$ and $|U_a| \leq 2^{n_u}$, the transition relation

$\xrightarrow[a]{}$ admits the alternative representation as a Boolean function $\delta_a : \mathbb{B}^{n_x} \times \mathbb{B}^{n_u} \times \mathbb{B}^{n_x} \to \mathbb{B}$, where:

$$\delta_a(\mathbf{b}_{n_x}(x), \mathbf{b}_{n_u}(u), \mathbf{b}_{n_x}(x')) = \textit{true} \Leftrightarrow (x, u, x') \in \overset{\cdots\cdots}{\underset{a}{\rightarrow}}$$

All the algorithms employed in the subsequent example have been implemented using ROBDD's to store the transition relations of every system involved in the algorithms.

B. Example

We illustrate the proposed technique on the classical example of the double integrator [5], where Σ is the control system:

$$\dot{\xi}_{x_0,\upsilon}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xi(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \upsilon(t)$$

and the target set W is the origin, *i.e.* $W = \{(0,0)\}$. In order to apply the proposed method one needs to enlarge the target set W.

Following the instructions presented in Section IV-C, first we select a precision $\varepsilon = 0.15$. Next we relax the problem by enlarging the target set to $W = \mathbf{B}_1((0,0))$. We select as parameters for the symbolic abstraction $\tau = 1$ and $\eta = 0.3$. Restricting the state set to $X = \mathbf{B}_{30}((0,0)) \subset \mathbb{R}^2$ the set $X_{\tau\eta}$ becomes finite and the proposed algorithms can be applied. Constructing $S_{\tau\eta}$ in Pessoa[1] over Matlab took less than 5 minutes and the resulting model required 7.9 MB to be stored. The lower bound required about 50 milliseconds while computing the time-optimal controller required only 3 seconds and the controller was stored in 1 MB.

We present the resulting bounds $J(S_c^* \times_{\mathcal{F}} S_{\tau\eta}, \lfloor W \rfloor_\eta, x_{\eta 0})$ and $J(S_{d(c)}^* \times_{\mathcal{F}} S_{d(\tau\eta)}, \lceil W \rceil_\eta, x_{\eta 0})$ for the cost function $J(S_{\tau c}^* \times_{\mathcal{F}} S_\tau, W, x_0)$ in Figure 1, and the approximately optimal controller S_c^* in Figure 2. Superimposed on Figure 2 is the switching curve for the optimal controller to reach the origin (as reported in [5]). It should be no surprise that this switching curve does not coincide with the one found by our toolbox, as the continuous controller is not optimal to reach the set W (it is just optimal when the target set is the singleton $\{(0,0)\}$). Although the computed bounds are conservative, the cost achieved with the symbolic controller is quite close to the true optimal cost. This is a consequence of the bounds relying entirely on the computed abstractions while the symbolic controller uses feedback from the real system. This is illustrated in Table I, in which the time to reach the target set W using the constructed controller is compared to the cost of reaching W with the optimal continuous controller to reach the origin.

VI. Discussion

We have proposed a computational approach to solve time-optimal control problems by resorting to abstractions of control systems that approximately simulate or alternatingly simulate the original control system. The solutions obtained provide explicit lower and upper bounds on the achievable cost. The techniques employed allows one to solve complex time-optimal control problems, with target sets, space sets and dynamics of very general nature. We have implemented the presented algorithms in a Toolbox for Matlab resorting to BDD's as the underlying data structures, and with them we generated an example to illustrate the proposed techniques. Future work will concentrate in the development of synthesis algorithms for combinations of qualitative and quantitative specifications for control systems.

[1] Pessoa is a software toolbox for the synthesis of correct-by-design embedded control software. Pessoa is scheduled to be made publicly available on November 2009.

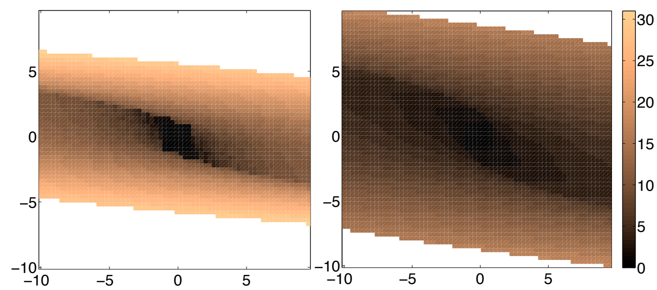

Fig. 1. Upper bound $J(S_c^* \times_{\mathcal{F}} S_{\tau\eta}, \lfloor W \rfloor_\eta, x_{\eta 0})$ (left) and lower bound $J(S_{d(c)}^* \times_{\mathcal{F}} S_{d(\tau\eta)}, \lceil W \rceil_\eta, x_{\eta 0})$ (right).

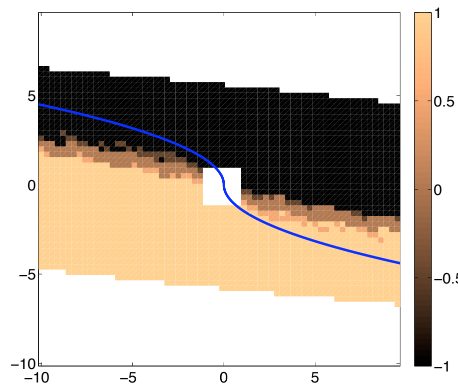

Fig. 2. Symbolic controller S_c^*.

Controller	$x_0 = (-6.1, 6.1)$	$(-6, 6)$	$(-5.85, 5.85)$
Continuous	12.83 s	12.66 s	11.60 s
Symbolic	14 s	14 s	13 s
UpperBound	29 s	29 s	29 s
LowerBound	9 s	9 s	9 s
Controller	$x_0 = (3.1, 0.1)$	$(3, 0)$	$(2.85, -0.1)$
Continuous	2.66 s	2.53 s	2.38 s
Symbolic	3 s	3 s	3 s
UpperBound	7 s	7 s	7 s
LowerBound	2 s	2 s	2 s

TABLE I

Times achieved in simulations by a continuous Sub-optimal controller and the symbolic controller.

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