Input-to-state stability of self-triggered control systems

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Abstract— **Event-triggered and self-triggered control have recently been proposed as an alternative to periodic implementations of feedback control laws over sensor/actuator networks. In event-triggered control, each sensing node continuously monitors the plant in order to determine if fresh information should be transmitted and if the feedback control law should be recomputed. In general, event-triggered control substantially reduces the number of exchanged messages when compared with periodic implementations. However, such energy savings must be contrasted with the energy required to perform local computations. In self-triggered control, computation of the feedback control law is followed by the computation of the next time instant at which fresh information should be sensed and transmitted. Since this time instant is computed as a function of the current state and plant dynamics, it is still much larger than the sampling period used in periodic implementations. Moreover, no energy is spent in local computations at the sensors. However, the plant operates in open-loop between updates of the feedback control law and robustness is a natural concern. We analyze the robustness to disturbances of a self-triggered implementation recently introduced by the authors for linear control systems. We show that such implementation is exponentially input-to-state stable with respect to disturbances.**

I. INTRODUCTION

A vast number of control systems are implemented nowadays on digital platforms. In those platforms, periodic implementations are prevalent due to the ease of design and analysis, and despite their inefficient use of resources. A system can be operating on a stable manifold of the state space and therefore require little or no attention, while if it lies on an unstable manifold, it might demand more control attention. Periodic implementations cannot take advantage of this fact because they disregard the information contained in the state. Moreover, the period is usually selected to guarantee a desired performance under worst case conditions even though these might rarely occur. With the advent of embedded and networked control systems, greater functionality is expected and control loops no longer have dedicated computational and communication resources at their disposal. While the implementation aspects were traditionally ignored at the design stage, this can no longer be done.

For these reasons event-triggered techniques for control, advocating the use (or update) of actuation only when certain events occur, have started to experience an increased attention in the literature. Many researchers have proposed diverse designs or implementations of controllers in the event-triggered form [1], [2], [3]. Recently, several implementations based on dissipation inequalities have been proposed, from the initial work of Tabuada [4] to the recent advances on implementations for distributed systems [5].

This surge of attention into event-triggered techniques have also lead to the proposal of the closely related idea of self-triggered control. A self-triggered implementation, as an event-triggered one, strives for reducing the amount of actuation necessary to stabilize the plant. While event-triggered techniques rely on continuous supervision of the plant, in order to detect a certain event to trigger actuation, self-triggered techniques aim precisely at removing the demand for continuous supervision (and thus measurement). First proposed by Velasco *et al* [6], the idea of self-triggered implementations is to select the next controller update instant based on the knowledge of the dynamics and the latest measurements of the plant state. Inspired by the previous work on Lyapunov based event-triggered control, several self-triggered implementations have been proposed, both for linear [7], [8] and non-linear [9] plants. Recently, the authors proposed in [10] a self-triggered implementation for linear state feedback controllers exhibiting times between controller updates much larger than those arising from periodic implementations.

When working with sensor/actuator networks for control applications, event-triggered and self-triggered control provide with reduced energy expenditures due to communication between the sensors and actuators by reducing actuation requirements. Moreover, self-triggered control provides additional energy savings for the sensor nodes by letting them go on sleep mode in between actuator updates. This is possible since the sensors are no longer responsible for detecting the event initiating a new controller update, as was the case with event-triggered control. On the other hand, self-triggered implementations work in open-loop in between controller updates, thus raising a natural concern about robustness against disturbances of such implementations.

The problem of reducing communication requirements on networked control systems was addressed by Nesic, Tabbara and Teel, with a different focus in [11] and [12]. Their work concentrates on finding the maximum time allowed to elapse between updates of the controller to retain stability. While their techniques account explicitly for the effect of the communication protocol in use, they provide with a uniform bound across the whole state-space, making their approach inherently periodic.

In this paper, we revisit the implementation introduced in [10] and study its robustness to disturbances. Following the theory of stability for sampled-data systems from [13], we prove that our implementation is exponentially input-to-state stable in the presence of disturbances. The work in [8] studies a similar problem, although there the disturbances are assumed to be bounded by a linear function of the norm

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II. Preliminaries

We denote by $\mathbb{R}^+$ to the positive real numbers. We also use $\mathbb{N}$ to denote the natural numbers including zero and $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$. The usual Euclidean ($l_2$) vector norm is represented by $|\cdot|$. When applied to a matrix $|\cdot|$ denotes the l_2 induced matrix norm. A matrix $P \in \mathbb{R}^{m\times m}$ is said to be positive definite, denoted $P > 0$, whenever $x^T P x > 0$ for all $x \neq 0$, $x \in \mathbb{R}^m$. By $\lambda_m(P), \lambda_M(P)$ we denote the minimum and maximum eigenvalues of P respectively. A function $\gamma : [0, a[\to \mathbb{R}_0^+$, $a > 0$ is of class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, $\gamma(0) = 0$ and $\gamma(s) \to \infty$ as $s \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times [0, a[\to \mathbb{R}_0^+$ is of class $\mathcal{KL}$ if $\beta(\cdot, \tau)$ is of class $\mathcal{K}_\infty$ for each $\tau \geq 0$ and $\beta(s, \cdot)$ is monotonically decreasing to zero for each $s \geq 0$. A class $\mathcal{KL}$ function $\beta(\tau, s)$ is called exponential if $\beta(s, \tau) \leq \sigma s e^{-c\tau}$, $\sigma > 0$, $c > 0$. Given an essentially bounded function $\delta : \mathbb{R}_0^+ \to \mathbb{R}^m$ we denote by $\|\delta\|_\infty$ the $\mathcal{L}_\infty$ norm, *i.e.* $\|\delta\|_\infty = \sup_{t\in\mathbb{R}_0^+}\{|\delta(t)|\} < \infty$.

In the following we will consider systems defined by differential equations of the form:

$$\frac{d}{dt}\xi = f(\xi) \tag{1}$$

in which $f : \mathbb{R}^m \to \mathbb{R}^m$ is a smooth map. We will also use the simpler notation $\dot{\xi} = f(\xi)$ to refer to (1). A solution or trajectory of (1) with initial condition $x \in \mathbb{R}^m$ is denoted by $\xi_x : \mathbb{R}_0^+ \to \mathbb{R}^m$, where ξ_x satisfies: $\xi_x(0) = x$ and $\frac{d}{dt}\xi_x(t) = f(\xi(t))$. We will relax the notation by dropping the x subindex whenever the initial condition is not relevant to the discussion.

We will also consider systems with inputs $\chi : \mathbb{R}_0^+ \to \mathbb{R}^l$ and $\delta : \mathbb{R}_0^+ \to \mathbb{R}^p$ essentially bounded piecewise continuous functions of time:

$$\frac{d}{dt}\xi = f(\xi, \chi, \delta) \tag{2}$$

in which $f : \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^p \to \mathbb{R}^m$ is a smooth map. We refer to such systems as *control systems*. Solutions of (2) with initial condition x and inputs χ and δ, denoted by $\xi_{x\chi\delta}$, satisfy: $\xi_{x\chi\delta}(0) = x$ and $\frac{d}{dt}\xi_{x\chi\delta}(t) = f(\xi_{x\chi\delta}(t), \chi(t), \delta(t)))$ for almost all $t \in \mathbb{R}_0^+$. As before, the notation will be relaxed by dropping the subindex when it does not contribute to the clarity of the exposition. The input χ will be used to denote controlled inputs, while δ will denote disturbances. A feedback law for a control system is a smooth map $g : \mathbb{R}^m \to \mathbb{R}^l$; we will sometimes refer to such a law as a *controller* for the system.

A function $V : \mathbb{R}^m \to \mathbb{R}_0^+$ continuous on $\mathbb{R}^m$ and smooth on $\mathbb{R}^m\backslash\{0\}$ is said to be a Lyapunov function for system (1) if $V(x) \geq 0$ for all $x \in \mathbb{R}^m$, $V(x) = 0$ implies $x = 0$, and there exists $\lambda \in \mathbb{R}^+$ such that for every $x \in \mathbb{R}^m\backslash\{0\}$:

$$\frac{\partial V}{\partial x} f(x) \leq -\lambda V(x).$$

We will refer to λ as the *rate of decay* of the Lyapunov function.

A continuous time system (1) is said to be *uniformly globally exponentially stable* (UGES) if there exists $\sigma, \lambda \in \mathbb{R}^+$ such that $\forall t \geq 0$ and $\forall x \in \mathbb{R}^m$:

$$|\xi_x(t)| \leq \sigma |x| e^{-\lambda t}.$$

In what follows, σ will be referred to as the *gain* and λ as the *rate of decay* of the UGES estimate.

Consider a control system described by the differential equation $\dot{\xi} = f(\xi, \delta)$. The system is said to be *uniformly input-to-state stable* (UISS) if there exists $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for any $t \geq 0$ and $\forall x \in \mathbb{R}^m$:

$$|\xi_x(t)| \leq \beta(|x|, t) + \gamma(\|\delta\|_\infty).$$

If β is also exponential, then the system is said to be *exponentially UISS*. We shall refer to (β, γ) as the *supply rates* of the UISS estimate.

III. Problem Statement

In this paper we analyze the robustness properties of a self-triggered implementation for linear state feedback controllers presented in [10]. More precisely we are interested in proving that such an implementation is *exponentially input-to-state stable*. Before formally introducing the problem we solve in this paper, it is convenient to define the notions of *Event-Triggered Control* and *Self-Triggered Control*.

A. Event-triggered control.

Consider the control system:

$$\begin{aligned} \dot{\xi} &= f(\xi, \chi) \\ \chi &= g(\xi) \end{aligned} \tag{3}$$

with the feedback law g rendering the closed loop exponentially stable. Assume now a sampled-data implementation:

$$\begin{aligned} \dot{\xi}(t) &= f(\xi(t), \chi(t)) \\ \chi(t) &= g(\xi(t_k)),\ t \in [t_k, t_{k+1}[\end{aligned} \tag{4}$$

An event-triggered implementation defines a sequence of update times $\{t_k\}$ for the controller, rendering the closed loop system asymptotically stable. This sequence of times is implicitly defined by a condition on the state of the plant, typically of the form:

$$h(t_k, \xi(t), \xi(t_k)) = 0 \quad \Rightarrow \quad t_{k+1} := t \tag{5}$$

where the map $h : \mathbb{R}_0^+ \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is continuous and at least differentiable on its second argument. Moreover h satisfies $\forall x_k \in \mathbb{R}^m$:

$$\begin{aligned} h(t_k, x_k, x_k) &= 0 \\ \frac{\partial h}{\partial x}(t_k, x, x_k) f(x, g(x))\Big|_{x=x_k} &< 0. \end{aligned} \tag{6}$$

These properties guarantee (by the continuity of h) that there exist a minimum time in between updates of the controller, *i.e.* for all $x_k \in \mathbb{R}^m$ there exists some $t_{min}(x_k) > 0$ such that $\tau_k = t_{k+1} - t_k \geq t_{min}(x_k)$, when $\xi(t_k) = x_k$. Also note that, as defined, h only takes negative values or the value zero, as h is reset to zero whenever the controller is updated. From here on, we will retain the notation τ_k to denote the inter-execution times.

B. Self-triggered control.

Self-triggered control removes the continuous supervision of conditions like (5), in favour of constructing maps $\tau_k = \Gamma(\xi(t_k))$ providing with the next controller update time as a function of the state when the last update was performed. If a condition of the form of (5) is available, one could attempt to design Γ so that $t \in]t_k, t_k+\Gamma(\xi(t_k))[\Rightarrow h(t_k, \xi(t), \xi(t_k)) < 0$. Such a design can be thought of as an emulation of the event-triggered implementation defined by (5).

Self-triggered controllers are less demanding in terms of sensing as measurements are only required at the $\{t_k\}$ instants. In systems where sensing, computation and actuation are not collocated, such as sensor/actuator networks, the reduced sensing requirements will also have an impact on the communication requirements between the different elements of the system and thus on its energy expenditures. Moreover, while event-triggered controllers require a continuous computation of h to test the condition triggering actuation, self-triggered controllers just perform computations when a new update of the controller takes place. Furthermore, in a wireless sensor/actuator network, reducing the listening time of the wireless nodes is of vital importance to extend their lifespan. In a self-triggered implementation, the nodes of the network are informed after each update of the time they can stay asleep until a next update is required. On the other hand, self-triggered control operates in open-loop in between controller updates (all the sensors are sleeping), which poses a natural concern in terms of robustness of these implementations. In the following sections we will address this concern.

C. Problem.

Now we are in position to formulate the question to which we devote the rest of the paper:

Problem 3.1: Consider a linear control system and controller rendering the closed loop exponentially stable:

$$\dot{\xi} = A\xi + B\chi \quad (7)$$
$$\chi = K\xi \quad (8)$$

Assume that a self-triggered implementation $\Gamma : \mathbb{R}^m \to \mathbb{R}^+$, $\Gamma(\xi(t_k)) = \tau_k$, retaining exponential stability of the system is available. In the presence of additive disturbances δ:

$$\dot{\xi} = A\xi + BK\xi(t_k) + \delta,\ t \in [t_k, t_k + \Gamma(\xi(t_k))[,$$

is the resulting closed-loop system *exponentially input-to-state stable*?

Remark 3.2: Additive disturbances can model both sensing and actuation disturbances. A more thorough analysis of a number of different phenomena encountered in practice that also fit in this model can be found in [14].

In what follows we give a positive answer to Problem 3.1 for the self-triggered implementation proposed by the authors in [10]. The self-triggered implementation under study is revisited in the following section.

IV. A SELF-TRIGGERED IMPLEMENTATION.

Let V be a Lyapunov function for the system (3) with rate of decay $\lambda_o \in \mathbb{R}^+$ and satisfying $\underline{\alpha}|x| \leq V(x) \leq \overline{\alpha}|x|$ for some $\underline{\alpha}, \overline{\alpha} \in \mathbb{R}^+$. Consider the system (4) with $\{t_k\}$ as the sequence of update times for the controller. Let now the event-triggered condition defining implicitly the controller update times be, for every $t > t_k$:

$$h(t_k, \xi(t), \xi(t_k)) := V(\xi(t)) - V(\xi(t_k))e^{-\lambda(t-t_k)} \quad (9)$$

for some $0 < \lambda < \lambda_o$. Such a condition forces the closed-loop system to satisfy for all $t \geq 0$, and for all $x_o \in \mathbb{R}^m$:

$$V(\xi_{x_o}(t)) \leq V(x_o)e^{-\lambda t}, \quad (10)$$

where the initial time $t_0 = 0$ is also assumed to coincide with the first execution of the controller. Note that (10) implies exponential stability of the closed-loop system in the absence of disturbances.

It can be verified that (9) satisfies the conditions in (6), and therefore there exists some $t_{min}(x_k)$ such that for all $x_k \in \mathbb{R}^m$, $\tau_k \geq t_{min}(x_k)$, when $\xi(t_k) = x_k$. In the linear case, where $V(x) = x^T Px,\ P > 0$, and $f(\xi, \chi) = A\xi + B\chi$, it was shown in [10] that there exist a uniform bound $t_{min} > 0$ such that $\tau_k \geq t_{min}$, and a numerical algorithm was provided to compute that lower bound for the inter-execution times.

So far we have just described a rather obvious event-triggered implementation that, in the absence of disturbances, provides exponential stability for the sampled-data implementation. We now review a self-triggered implementation for linear systems based on an emulation of the event-triggered implementation just presented. The implementation tests condition (9) at discrete instants of time separated by Δ units of time from each other. Making use of an exact discrete-time model, (9) can be evaluated only from the knowledge of the initial condition. For efficiency of storage and evaluation the technique uses a representation of the state in a higher dimensional space in which (9) becomes a linear condition. We describe this process in the following paragraphs.

Let us start defining:

$$N_{min} := \left\lfloor \frac{t_{min}}{\Delta} \right\rfloor,\ N_{max} := \left\lfloor \frac{t_{max}}{\Delta} \right\rfloor$$

where t_{min} is the minimum time between updates, implicitly defined by our choice of λ, and t_{max} the maximum time between updates, selected as a design parameter. The selection of t_{min} (through the design of λ) and t_{max} will define the performance of the system as the analysis in the next section shows.

The exact discrete time model of system (7) with discretization step Δ is given by:

$$\begin{aligned}
\xi(t_k + (n+1)\Delta) &= A_d\xi(t_k + n\Delta) + B_d\chi(t_k + n\Delta), \\
A_d &:= e^{A\Delta}, \\
B_d &:= \int_0^\Delta e^{A(\Delta-\tau)}Bd\tau
\end{aligned}$$

with $n \in [N_{min}, N_{max}]$. With this discrete model and letting $\chi(t_k + n\Delta) = Kx_k$, $x_k := \xi(t_k)$ one can compute:

$$\begin{aligned} \xi(t_k + n\Delta) &= R(n)\xi(t_k) \\ R(n) &:= A_d^n + \sum_{i=0}^{n-1} A_d^i B_d K \end{aligned}$$

The function $W(x) := x^T P x$, in the absence of disturbances, would take the value:

$$W(\xi(t_k + n\Delta)) = \xi^T(t_k) R^T(n) P R(n) \xi(t_k).$$

Testing condition (9) transforms into finding the largest value of n such that:

$$\xi^T(t_k) \left(R^T(n) P R(n) - P e^{-2\lambda(n\Delta)} \right) \xi(t_k) \leq 0.$$

Note that if the selected Lyapunov function is $V(x) = (x^T P x)^{\frac{1}{2}}$, the above procedure also ensures that $\forall n \in \mathbb{N}$:

$$V(\xi(t_k + n\Delta)) \leq V(\xi(t_k)) e^{-\lambda(n\Delta)}.$$

The self-triggered algorithm proposed in [10] can now be summarized in the following theorem:

Theorem 4.1: Given the linear control system:

$$\begin{aligned} \dot{\xi}(t) = A\xi(t) + BK\xi(t_k) + \delta(t), \\ t \in [t_k, t_k + \Gamma(\xi(t_k))[, \end{aligned} \tag{11}$$

with the self-triggered implementation $\Gamma : \mathbb{R}^m \to \mathbb{R}^+$, $\Gamma(\xi(t_k)) = \tau_k$, determined by the policy:

$$\begin{aligned} \Gamma(\xi(t_k)) &:= \max\{t_k + t_{min}, t_k + \tilde{n}_k \Delta\}, \\ \tilde{n}_k &:= \max\{s \leq N_{max} | \, \tilde{h}(n, \xi(t_k)) \leq 0, \forall n \in [0, s]\}, \\ \tilde{h}(n, x_k) &:= |P^{\frac{1}{2}} R(n) x_k| - V(x_k) e^{-\lambda(n\Delta)} \end{aligned}$$

is *exponentially input-to-state stable*.

The proof of this result is deferred to the next section in which we discuss the complexity of this implementation, stability bounds and robustness measures as functions of λ, t_{min}, Δ and N_{max}. Note that λ, Δ and N_{max} are design parameters, whereas the value of t_{min} is determined by that of λ. Also note, how this technique could be transformed into an event-triggered form by taking measurements every Δ units of time. However, such event-triggered approach will suffer from a communication burden in systems with distributed sensing.

V. Analysis.

Consider the system (4) with zero-order-hold in the controlled input and with $\{t_k\}$ as the sequence of update times for the controller. Let also $V(x)$ be a Lyapunov function for the system (4) with rate of decay $\lambda_o \in \mathbb{R}^+$. Assume in addition that for all $k \in \mathbb{N}$ the inter-execution times $\tau_k \in [N_{min}\Delta, N_{max}\Delta]$.

The following result extends Theorem 4 in [13] to the case in which estimates (or measurements) of the state are available more often than the controller is updated. Let the estimates of the state be available every Δ units of time, while actuation occurs at instants separated by irregular intervals of length τ_k.

Theorem 5.1: Consider the system (4). If the following two conditions are satisfied:

- there exist some $L_1, L_2 \in \mathbb{R}_0^+$ such that:

$$|\xi(t + \tau)| \leq L_1 |\xi(t)| + L_2 |\xi(t_k)|,$$

for all $t \in [t_k, t_{k+1}]$ and $\tau \in [0, \Delta]$,

- there exist $\sigma, \lambda \in \mathbb{R}^+$ such that $\forall k \geq 0$ and $\forall x \in \mathbb{R}^m$:

$$|\xi(t_k)| \leq \sigma |\xi(t_0)| e^{-\lambda k},$$

then the sampled data system is UGES with decay rate λ and gain $g(\Delta, N_{max})$ given by:

$$g(\Delta, N_{max}) := \sigma L_1 e^{\lambda\Delta} + L_2 e^{\lambda N_{max}\Delta}$$

Proof: From the assumptions we can directly conclude that:

$$\begin{aligned} |\xi(t_k + \tau)| \leq L_1 |\xi(t_k)| e^{-\lambda n\Delta} + L_2 |\xi(t_k)|, \\ \forall t_k \geq t_0, \ \forall \tau \in [n\Delta, (n+1)\Delta], \ n + 1 \leq N_{max} \end{aligned}$$

At every instant $t_k + n\Delta$, $n \in [0, N_{max}]$ and for all $t_k \geq t_0$ we know that $|\xi(t_k + n\Delta)| \leq \sigma |\xi(t_k)| e^{-\lambda\Delta n}$, but we ignore the value of the Lyapunov function in between measurements. Nonetheless, we can bound $|\xi(t_0 + r)|$ for all $r \geq 0$ as a function of $S(t) = |\xi(t_0)| e^{-\lambda(t - t_0)}$ by computing the following ratio:

$$g(\Delta, N_{max}) := \max_r \frac{|\xi(t_0 + r)|}{S(t_0 + r)}$$

Which can be obtained bounding:

$$\begin{aligned} \frac{|\xi(t_0 + r)|}{|\xi(t_0)| e^{-\lambda r}} \leq \frac{\sigma L_1 |\xi(t_k)| e^{-\lambda n\Delta} + L_2 |\xi(t_k)|}{|\xi(t_k)| e^{-\lambda(n\Delta + s)}}, \\ r = t_k + n\Delta + s - t_0, \ n \in [0, (N_{max} - 1)], \ s \in [0, \Delta] \end{aligned}$$

and then taking the maximum over all $r > 0$, *i.e.* over all $n \in [0, (N_{max} - 1)], \ s \in [0, \Delta]$:

$$\frac{|\xi(t_0 + r)|}{S(t_0 + r)} \leq \sigma L_1 e^{\lambda\Delta} + L_2 e^{\lambda N_{max}\Delta}$$

The proof is concluded by noting that the resulting expression is independent of k. ■

Remark 5.2: The values of both L_1 and L_2 will in general depend on the selected value of Δ, normally decreasing for smaller values of Δ.

Remark 5.3: The previous theorem essentially differs from the result in [13] on the β function provided for the decay of the sampled-data system. The bound we provide has a smaller gain than will result from applying Theorem 4 of [13] with $T = N_{max}\Delta$ at the cost of more frequent measurements (estimates in our implementation).

Let us introduce now some new notation for succinctness of the expressions that appear in the rest of the paper. We will denote by ρ_P and $\tilde{\rho}_P$ to the following quantities:

$$\rho_P := \left(\frac{\lambda_M(P)}{\lambda_m(P)} \right)^{\frac{1}{2}}, \ \tilde{\rho}_P := \rho_P \lambda_M^{\frac{1}{2}}(P).$$

Corollary 5.4: Consider system (7), (8) admitting the Lyapunov function $V(x) = (x^T P x)^{\frac{1}{2}}$, under the self-triggered implementation presented in section IV with $\lambda \leq$

λ_o and Δ as discretization step. Such an implementation is UGES with decay rate λ and gain $g(\Delta, N_{max})$ where:

$$\begin{aligned} g(\Delta, N_{max}) &:= \rho_P L_1 e^{\lambda\Delta} + L_2 e^{\lambda N_{max}\Delta}, \\ L_1 &:= \max_{\tau\in[0,\Delta]} |e^{A\tau}|, \\ L_2 &:= \max_{\tau\in[0,\Delta]} \Big|\int_0^\tau e^{A(\tau-s)}BKds\Big|. \end{aligned}$$

Proof: The result is obtained by direct application of Theorem 5.1. The first condition is satisfied with L_1 and L_2 as defined by any linear system; the second condition is satisfied from the guarantee (10) provided by the implementation in section IV. ■

Actually, linear systems exhibit more properties that allow to obtain tighter bounds than the ones provided by the previous theorem, see [10].

Consider now the system:

$$\begin{aligned} \dot{\xi}(t) &= A\xi(t) + B\chi(t) + \delta(t) \\ \chi(t) &= K\xi(t_k),\ t \in [t_k, t_k + \Gamma(\xi(t_k))) \end{aligned} \quad (12)$$

where δ represents a disturbance affecting the system and Γ given by the implementation presented in section IV.

We introduce now a Lemma that we will use in the proof of Theorem 5.6.

Lemma 5.5: Let

$$\dot{\xi}(t) = A\xi(t) + B\chi(t) + \delta(t)$$

and $V(x) := (x^T P x)^{\frac{1}{2}}$. Then for any given $0 \le T < \infty$:

$$V(\xi_{x\chi\delta}(t)) \le V(\xi_{x\chi 0}(t)) + \gamma(\|\delta\|_\infty), \forall t \in [0, T]$$

with $\gamma \in \mathcal{K}_\infty$:

$$\gamma(s) = s\tilde{\rho}_P \int_0^T |e^{Ar}|dr$$

Proof: From the Lipschitz continuity of V we have

$$\begin{aligned} V(\xi_{x\chi\delta}(t)) &= |V(\xi_{x\chi 0}(t)) + V(\xi_{x\chi\delta}(t)) - V(\xi_{x\chi 0}(t))| \\ &\le V(\xi_{x\chi 0}(t)) + \tilde{\rho}_P|\xi_{x\chi\delta}(t) - \xi_{x\chi 0}(t)|. \end{aligned} \quad (13)$$

Integrating the dynamics of ξ and after applying Hölder's inequality one can conclude that:

$$|\xi_{x\chi\delta}(t) - \xi_{x\chi 0}(t)| \le \int_0^t |e^{Ar}|dr\|\delta\|_\infty \le \int_0^T |e^{Ar}|dr\|\delta\|_\infty \quad (14)$$

for all $t \in [0, T]$. Combining (13) and (14) concludes the proof. ■

In the following theorem we use the following notation in order to provide more succinct expressions:

$$\gamma_I(s) := s\int_0^{N_{max}\Delta} |e^{Ar}|dr,\ \gamma_P(s) := \tilde{\rho}_P\gamma_I(s). \quad (15)$$

Theorem 5.6: System (12) with Γ given by the self-triggered implementation in Theorem 4.1 is exponentially UISS with supply rates (β, γ):

$$\begin{aligned} \beta(s,t) &:= \rho_P g(\Delta, N_{max})e^{-\lambda t}s, \\ \gamma(s) &:= \gamma_I(s)\left(\frac{\rho_P^2 g(\Delta, N_{max})}{1 - e^{-\lambda t_{min}}} + 1\right) \end{aligned}$$

Proof: From Lemma 5.5, and the condition enforced by the self-triggered implementation we have:

$$V(\xi(t_{k+1})) \le V(\xi(t_k))e^{-\lambda\tau_k} + \gamma_P(\|\delta\|_\infty).$$

Iterating the previous equation it follows:

$$\begin{aligned} V(\xi(t_k)) &\le e^{-\lambda(t_k - t_0)}V(\xi(t_0)) + \gamma_P(\|\delta\|_\infty)\sum_{i=0}^{k-1} e^{-\lambda t_{min} i} \\ &\le e^{-\lambda(t_k - t_0)}V(\xi(t_0)) + \gamma_P(\|\delta\|_\infty)\sum_{i=0}^{\infty} e^{-\lambda t_{min} i} \\ &= e^{-\lambda(t_k - t_0)}V(\xi(t_0)) + \gamma_P(\|\delta\|_\infty)\frac{1}{1 - e^{-\lambda t_{min}}}. \end{aligned}$$

Assuming without loss of generality that $t_0 = 0$, the following bound also holds:

$$|\xi_x(t_k)| \le \rho_P|x|e^{-\lambda t_k} + \lambda_m^{-\frac{1}{2}}(P)\frac{\gamma_P(\|\delta\|_\infty)}{1 - e^{-\lambda t_{min}}} \quad (16)$$

From Theorem 5.1 and Lemma 5.5 one also obtains that for all $\tau \in [0, T]$:

$$|\xi_x(t_k + \tau)| \le g(\Delta, N_{max})|\xi_x(t_k)|e^{-\lambda\tau} + \gamma_I(\|\delta\|_\infty). \quad (17)$$

Combining (16) and (17) results in:

$$\begin{aligned} |\xi_x(t_k + \tau)| &\le g(\Delta, N_{max})\rho_P|x|e^{-\lambda(t_k+\tau)} + \gamma_I(\|\delta\|_\infty) \\ &+ e^{-\lambda\tau}\gamma_P(\|\delta\|_\infty)\frac{\lambda_m^{-\frac{1}{2}}(P)g(\Delta, N_{max})}{1 - e^{-\lambda t_{min}}} \end{aligned}$$

and after denoting $t = t_k + \tau$ we can further bound:

$$\begin{aligned} |\xi_x(t)| &\le g(\Delta, N_{max})\rho_P|x|e^{-\lambda t} \\ &+ \gamma_P(\|\delta\|_\infty)\frac{\lambda_m^{-\frac{1}{2}}(P)g(\Delta, N_{max})}{1 - e^{-\lambda t_{min}}} + \gamma_I(\|\delta\|_\infty), \end{aligned}$$

which is independent of k and concludes the proof. ■

VI. EXAMPLE.

To illustrate the performance of the proposed self-triggered implementation we borrow the Batch Reactor model from [15] with state space description:

$$\dot{\xi} = \begin{bmatrix} 1.38 & -0.20 & 6.71 & -5.67 \\ -0.58 & -4.29 & 0 & 0.67 \\ 1.06 & 4.27 & -6.65 & 5.89 \\ 0.04 & 4.27 & 1.34 & -2.10 \end{bmatrix}\xi + \begin{bmatrix} 0 & 0 \\ 5.67 & 0 \\ 1.13 & -3.14 \\ 1.13 & 0 \end{bmatrix}\chi$$

A state feedback controller placing the poles of the closed loop system at $\{-3 + 1.2i, -3 - 1.2i, -3.6, -3.9\}$ is:

$$K = \begin{bmatrix} 0.1006 & -0.2469 & -0.0952 & -0.2447 \\ 1.4099 & -0.1966 & 0.0139 & 0.0823 \end{bmatrix}$$

The closed loop has as decay rate $\lambda_o = 0.41$ and we set $\lambda = 0.9\lambda_o$. The resulting minimum time for this selection of λ is $t_{min} = 18ms$. The rest of design values were set to: $t_{max} = 358ms$ and $\Delta = 10ms$. With this design the complexity becomes $M_s = 350$ and $M_t = 745$.

Figure 1 presents the evolution of $V(\xi(t))$ (solid line) and the piecewise continuous function $V(\xi(t_k))e^{-\lambda(t-t_k)}$ (dotted line) between seconds 1 and 2. The intersection of the dotted and solid lines (or the maximum value $N_{max}\Delta$) determines the inter-execution times τ_k as defined in Theorem 4.1. The

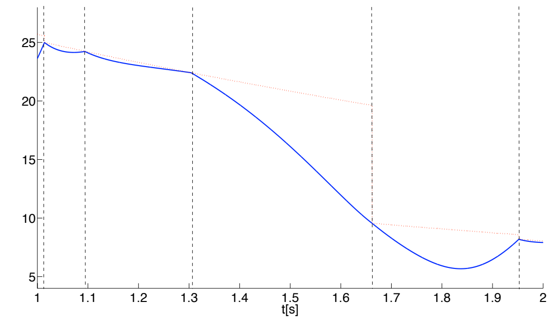

Fig. 1. $V(\xi(t))$ (solid line) and $V(\xi(t_k))e^{-\lambda(t-t_k)}$ (dotted line) illustrating the triggering of new actuation.

actuation times t_k are marked with vertical dashed lines in Figure 1.

Figure 2 depicts the inter-execution times τ_k generated by the self-triggered implementation in the absence of disturbances. The evolution of the Lyapunov function

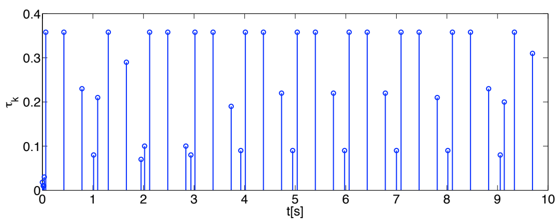

Fig. 2. Inter-execution times when no disturbance is present.

$V(x) = (x^T P x)^{\frac{1}{2}}$ under disturbances (uniformly distributed bounded noise) with norms $\|\delta\|_\infty = 1$ and $\|\delta\|_\infty = 10$ are presented in figures 3, and 5 respectively. In both of those figures the ISS nature of the system can be appreciated. We also present the inter-execution times τ_k generated by the self-triggered implementation under the presence of a disturbance with $\|\delta\|_\infty = 1$ in figure 4.

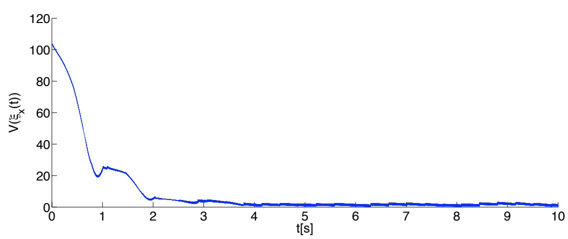

Fig. 3. Lyapunov function evolution under the presence of a disturbance of norm $\|\delta\|_\infty = 1$.

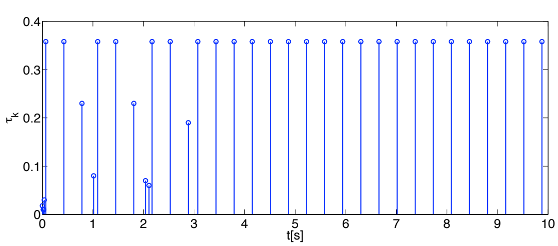

Fig. 4. Inter-execution times under the presence of a disturbance of norm $\|\delta\|_\infty = 1$.

VII. Discussion.

We proved that the self-triggered implementation introduced in [10] is UISS and provided with estimates for

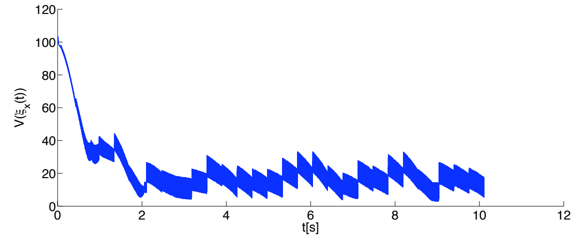

Fig. 5. Lyapunov function evolution under the presence of a disturbance of norm $\|\delta\|_\infty = 10$.

the supply rates (β, γ). As expected, the effect of disturbances on the performance guarantees grows as one increases $N_{max}\Delta$ (the maximum time the system is allowed to run in open-loop). Moreover, the explicit bounds of the effect of the quantization interval Δ on the pair (β, γ) (through $g(\Delta, N_{max})$) and on the complexity of the implementation provide guidelines for practical designs of self-triggered controllers.

Although all the present paper is devoted to linear control systems, the application of similar ideas to non-linear systems (via approximate models) is on the scope of future research.

References

[1] K. Åström and B. Bernhardsson, "Comparison of Riemann and Lebesgue sampling for first order stochastic systems," *Proceedings of the 41st IEEE Conference on Decision and Control*, vol. 2, pp. 2011–2016, 2002.

[2] K. Årzén, "A simple event based PID controller," *Proceedings of 14th IFAC World Congress*, vol. 18, pp. 423–428, 1999.

[3] W. Heemels, J. Sandee, and P. van den Bosch, "Analysis of event-driven controllers for linear systems," *Int. J. of Control*, pp. 81(4), 571–590, 2008.

[4] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Transactions on Automatic Control*, vol. 52(9), pp. 1680–1685, 2007.

[5] M. Mazo Jr. and P. Tabuada, "On event-triggered and self-triggered control over sensor/actuator networks," *Proceedings of the 47th IEEE Conference on Decision and Control*, pp. 435–440, 2008.

[6] M. Velasco, J. Fuertes, and P. Marti, "The self triggered task model for real-time control systems," *Work in Progress Proceedings of the 24th IEEE Real-Time Systems Symposium*, pp. 67–70, 2003.

[7] M. Lemmon, T. Chantem, X. Hu, and M. Zyskowski, "On Self-Triggered Full Information H-infinity Controllers," *Hybrid Systems: Computation and Control, April*, 2007.

[8] X. Wang and M. Lemmon, "Self-triggered Feedback Control Systems with Finite-Gain L2 Stability," *IEEE Transactions on Automatic Control*, vol. 45(3), pp. 452–467, 2009.

[9] A. Anta and P. Tabuada, "To sample or not to sample: Self-triggered control for nonlinear systems," *Arxiv preprint arXiv:0806.0709*, Jan 2008. [Online]. Available: http://arxiv.org/pdf/0806.0709

[10] M. Mazo Jr., A. Anta, and P. Tabuada, "On self-triggered control for linear systems: Guarantees and complexity," *European Control Conference*, 2009.

[11] D. Nesic and A. Teel, "Input-output stability properties of networked control systems," *IEEE Transactions on Automatic Control*, vol. 40(10), pp. 1650–1667, Jan 2004.

[12] M. Tabbara, D. Nesic, and A. Teel, "Input-output stability of wireless networked control systems," *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC '05. 44th IEEE Conference on*, pp. 209– 214, 2005.

[13] D. Nesic, A. Teel, and E. Sontag, "Formulas relating KL stability estimates of discrete-time and sampled-data nonlinear systems," *Systems & Control letters*, vol. 38, pp. 49–60, Jan 1999.

[14] C. Kellett, H. Shim, and A. Teel, "Further results on robustness of (possibly discontinuous) sample and hold feedback," *IEEE Transactions on Automatic Control*, vol. 9(7), pp. 1081–1089, Jan 2004.

[15] G. C. Walsh and H. Ye, "Scheduling of Networked Control Systems," *IEEE Control Systems Magazine*, pp. 57–65, February 2001.