Timing Abstraction of Perturbed LTI systems with $\mathcal{L}_2$-based Event-Triggering Mechanism

Arman Sharifi Kolarijani, Manuel Mazo Jr. and Tamás Keviczky

***Abstract*— In networked control systems, the advent of event-triggering strategies in the sampling process has resulted in the usage reduction of network capacities, such as communication bandwidth. However, the aperiodic nature of sampling periods generated by event-triggering strategies has hindered the schedulability of such networks. In this study, we propose a framework to construct a timed safety automaton that captures the sampling behavior of perturbed LTI systems with an $\mathcal{L}_2$-based triggering mechanism proposed in the literature. In this framework, the state-space is partitioned into a finite number of convex polyhedral cones, each cone representing a discrete mode in the abstracted automaton. Adopting techniques from stability analysis of retarded systems accompanied with a polytopic embedding of time, LMI conditions to characterize the sampling interval associated with each region are derived. Then, using reachability analysis, the transitions in the abstracted automaton are derived.**

I. Introduction

Wireless networked controlled systems (WNCS's) represent a class of spatially distributed control systems for which the feedback loops are closed via shared communication components possessing limited bandwidth. Several advantages of WNCSs, such as their ease of maintenance and flexibility of implementation, make them attractive to industrial environments. Meanwhile, WNCS's are burdened with characteristics, such as limited battery life and communication bandwidth. Under these circumstances, the resource over-utilization caused by (traditional) periodic implementations, the so-called time-driven control (TDC), makes such implementations less appealing for WNCS's.

To address the aforementioned issues, control researchers have proposed *event-driven control* (EDC) strategies that are aperiodic, such as *event-triggered control* (ETC) [1] and *self-triggered control* (STC) [2]. In EDC strategies, the dynamics of the control system during the inter-sample interval determine the next sampling instant to attenuate the usage of resources, particularly the communication bandwidth. In these strategies, control task executions only happen when a pre-specified condition is violated. Such condition is called the *triggering mechanism* (TM). It is derived based on stability and/or performance of the closed-loop system. On the other hand, the *schedulability* of ETC strategies, due to their aperiodic nature, is more arduous compared to TDC strategies. In fact, in TDC strategies, the control and scheduler designs are naturally decoupled via the (pre-defined) fixed sampling period. This phenomenon is called the *separation-of-concerns* in the real-time systems community [3]. ETC strategies are almost always equipped with a *minimum inter-execution time* (MIET) to prevent the occurrence of *Zeno* behavior in the sampling process. The MIET can be technically used in the synthesis of task scheduling. However, it is a coarse lower approximation of all the possible generated sampling periods. Thus, such synthesis does not make use of the beneficiary characteristics of ETC strategies in an efficient manner. To address this shortcoming, researchers have proposed another class of approaches, the so-called *co-design* approaches. In this class, the problem of controller and scheduler synthesis for real-time systems is tackled in a unified framework, see e.g. feedback modification to task attributes [4], [5], anytime controllers [6], [7], and event-based control and scheduling [8], [9]. Alternative to the unified frameworks mentioned above, [10], [11] have proposed a decoupling framework to capture the sampling behavior of LTI systems with ISS-based TM's using timed safety automata (TSA's). Generally speaking, TSA is a simplified version of timed automaton (TA) [12], [13]. It is a powerful tool to model the timing behavior of real-time systems for scheduling purposes [14].

In this study, following the same path as in [10], [11], we propose a framework to capture the sampling behavior of perturbed LTI systems with the $\mathcal{L}_2$-based TM proposed by [15]. The framework constructs a TSA that ε-approximately simulates the sampling behavior of the $\mathcal{L}_2$-based ETC system of [15]. The derived TSA can be analyzed independently for scheduling purposes, thus providing a scalable and versatile event-triggered WNCS design procedure. Due to space limitations, the proofs of Lemmas 2 & 3 and Theorems 2 & 3 and are omitted in this paper and can be found in [16].

The authors are with the Delft Center for Systems and Control, Delft University of Technology, The Netherlands.
{a.sharifikolarijani,m.mazo,t.keviczky}@tudelft.nl

II. Preliminaries

$\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^+$ denotes the positive reals. $\mathbb{N}_{>0}$ and $\mathbb{N}_0$ are the sets of positive integers and nonnegative integers, and $\mathbb{IR}^+$ is the set of all closed intervals $[a, b]$ such that $a, b \in \mathbb{R}^+$ and $a \leq b$. For any set S, 2^S denotes the set of all subsets of S, i.e. the power set of S. $\mathcal{S}_{m\times n}$ and $\mathcal{S}_n$ are the set of all $m \times n$ real-valued matrices and the set of all $n\times n$ real-valued symmetric matrices, respectively. For a matrix M, $M \preceq 0$ (or $M \succeq 0$) means M is a negative (or positive) semidefinite matrix and $M \prec 0$ ($M \succ 0$) indicates M is a negative (positive) definite matrix. $\mathcal{S}_n^+$ is the cone of all $n\times n$ symmetric positive definite matrices. $\lfloor x \rfloor$ indicates the largest integer not greater than $x \in \mathbb{R}$. $|y|$ and $\|M\|$ denote the Euclidean norm of a vector $y \in \mathbb{R}^n$ and the Frobenius norm of a matrix $M \in \mathcal{S}_{m\times n}$, respectively. For a matrix $M \in \mathcal{S}_n$, $\lambda(M)$ and $\lambda_{\max}(M)$ denote the set of eigenvalues and the largest eigenvalue of M. Consider two sets $X, Y \subseteq \mathbb{R}^n$, their Minkowski sum is given by $X \oplus Y := \{x + y | x \in X \text{ and } y \in Y\}$.

A. $\mathcal{L}_2$-Based ETC System:

Here, an overview of the ETC strategy proposed by [15] along with a new result (see Theorem 1) are presented. Consider a sampled-data system that is given by:

$$\dot{\xi}(t) = A\xi(t) + B\nu(t) + E\omega(t), \forall t \in [0, \tau(x)), \xi(0) = x, \tag{1}$$

where $\xi(t) \in \mathbb{R}^n$, $\nu(t) \in \mathbb{R}^m$, $\omega(t) \in \mathbb{R}^p$, $\tau(x)$ denotes the sampling period associated with $\xi(0)$, and A, B, and E have compatible dimensions. The control law is implemented in a sample-and-hold manner as follows:

$$\nu(t) = -Kx. \tag{2}$$

Assume that ω is a vanishing type disturbance [15], i.e.,

$$\exists W \geq 0 \text{ such that } |w(t)|^2 \leq W|x|^2, \forall t \in [0, \tau(x)). \tag{3}$$

Denote by ϵ, the error signal endured by the system (1)-(2), $\epsilon(t) = x - \xi_x(t)$ where $\xi_x(t)$ is the solution of (1). The evolution of state and error can be reformulated into the following compact forms:

$$\xi_x(t) = \Lambda(t)x + \Omega(t), \tag{4}$$

$$\epsilon(t) = (I - \Lambda(t))\, x - \Omega(t) \tag{5}$$

where

$$\begin{cases} \Lambda(t) = I + \int_0^t e^{As} ds (A - BK), \\ \Omega(t) = \int_0^t e^{A(t-s)} E\omega(s) ds. \end{cases} \tag{6}$$

Assume that there exists a quadratic Lyapunov function $V(\xi) = \xi^T P \xi$ such that P is the solution to the Algebraic Riccati Equation (ARE) given by:

$$PA + A^T P - Q + R = 0 \tag{7}$$

where

$$Q = PBB^T P, \quad R = \frac{1}{\gamma^2} PEE^T P, \quad \gamma > 0. \tag{8}$$

The existence of V guarantees that (1) with $\nu(t) = -K\xi(t) = -B^T P \xi(t)$ is $\mathcal{L}_2$ stable from ω to (x^T, u^T) with a finite-gain less than γ [15]. Then, consider a user-defined scalar $\beta > 0$, and matrices M and N defined in (9), the state-dependent TM proposed by [15] is given by (10).

$$M = (1-\beta^2)I + PBB^T P, N = \frac{1}{2}(1-\beta^2)I + PBB^T P \tag{9}$$

$$\tau(x) := \inf\{t > 0 |\ \epsilon^T(t) M \epsilon(t) \geq x^T N x\} \tag{10}$$

Theorem 1: Consider the system (1)-(2) with the triggering mechanism (10). Assume there exist a scalar μ and a symmetric matrix Ψ such that

$$\mu \geq 0, \quad \Psi \succ 0, \quad M + \Psi \preceq \mu I, \tag{11}$$

$$\Phi(t) \succeq 0, \tag{12}$$

are satisfied where

$$\Phi(t) = \begin{bmatrix} \Phi_1(t) & \Phi_2(t) \\ \Phi_2^T(t) & \Phi_4(t) \end{bmatrix}, \tag{13}$$

$$\begin{aligned} \Phi_1(t) = (\Lambda(t) - I)^T M (\Lambda(t) - I) \\ + tW\mu\lambda_{\max}(E^T E) d_A(t) I - N, \\ \Phi_2(t) = (\Lambda(t) - I)^T M^T,\ \Phi_4(t) = -\Psi. \end{aligned}$$

Then, $\tau(x)$ generated by the TM (10) is lower bounded by:

$$\tau'(x) := \inf\{t > 0 |\ \Phi(t) \succeq 0\}. \tag{14}$$

Proof: Using (5) the TM (10) can be reformulated into (15) where $\mathcal{F}_\omega(x,t)$ is given by (16):

$$\tau(x) = \min\{t > 0 |\ \mathcal{F}_\omega(x,t) \geq 0\}, \forall x \in \mathbb{R}^n \tag{15}$$

$$\begin{aligned} \mathcal{F}_\omega(x,t) &= x^T[(\Lambda(t) - I)^T M (\Lambda(t) - I) - N]x \\ &+ x^T(\Lambda(t) - I)^T M \Omega(t) + \Omega^T(t) M (\Lambda(t) - I) x \\ &+ \Omega^T(t) M \Omega(t) \end{aligned} \tag{16}$$

Let $\lambda_{\max}^A$ denote $\lambda_{\max}(A + A^T)$ for the sake of compactness. Using [16, Lemma 1] the terms that are dependent on both x and $\Omega(t)$ in $\mathcal{F}_\omega(x,t)$ can be decoupled into:

$$\begin{aligned} & x^T(\Lambda(t) - I)^T M \Omega(t) + \Omega^T(t) M (\Lambda(t) - I) x \leq \\ & \Omega^T(t) \Psi \Omega(t) + x^T (\Lambda(t) - I)^T M \Psi^{-1} M (\Lambda(t) - I) x, \end{aligned} \tag{17}$$

where $\Psi = \Psi^T \succ 0$. Then, it follows that:

$$\begin{aligned} & \Omega^T(t)(M + \Psi)\Omega(t) \\ & \leq \mu (\textstyle\int_0^t e^{A(t-s)} E\omega(s) ds)^T (\int_0^t e^{A(t-s)} E\omega(s) ds) \\ & \text{(assuming } M + \Psi \preceq \mu I \text{ and } \mu \geq 0) \\ & \leq t\mu \textstyle\int_0^t e^{(t-s)\lambda_{\max}^A} \omega^T(s) E^T E \omega(s) ds \\ & \text{(using [16, Proposition 1 and Lemma 2])} \\ & \leq tW\mu\lambda_{\max}(E^T E)(\textstyle\int_0^t e^{\lambda_{\max}^A (t-s)} ds)|x|^2 \\ & \text{(using (3))} \\ & = tW\mu\lambda_{\max}(E^T E) d_A(t) x^T x, \end{aligned} \tag{18}$$

where

$$d_A(t) = \begin{cases} \frac{1}{\lambda_{\max}^A}(e^{\lambda_{\max}^A t} - 1), \lambda_{\max}^A \neq 0 \\ t, \lambda_{\max}^A = 0. \end{cases} \tag{19}$$

Based on the aforementioned procedure, one derives the inequality in (20) where $\Theta(t)$ is given by (21).

$$\mathcal{F}_\omega(x,t) \leq x^T \Theta(t) x \tag{20}$$

$$\begin{aligned} \Theta(t) &= (\Lambda(t) - I)^T (M + M\Psi^{-1}M)(\Lambda(t) - I) \\ &+ tW\mu\lambda_{\max}(E^T E) d_A(t) I - N \end{aligned} \tag{21}$$

Then, the Schur complement is used to transform (21) into (13). Note that (21) is nonlinear in Ψ while (13) is linear in Ψ. Considering (20), since $\Phi(t) \succeq 0$ implies $x^T \Theta(t) x \geq 0$ by the Schur complement, it follows that $\tau(x) \geq \tau'(x)$. This concludes the proof. ■

Thus, Theorem 1 circumvents unknown effects of $\omega(t)$ in analyzing (10). However, (14) is still continuously dependent on t and lacks any information between x and τ'.

B. Systems and Relations

In what follows, we review some notions from the field of system theory to formally characterize the outcome of the proposed framework. Let Z be a set and $Q \subseteq Z \times Z$ be an equivalence relation on Z. Then, $[z]$ denotes the equivalence class of $z \in Z$ and Z/Q denotes the set of all equivalence classes. A metric (or a distance function) $d : Z \times Z \to \mathbb{R} \cup \{+\infty\}$ on Z satisfies, $\forall x, y, z \in Z$: i) $d(x,y) = d(y,x)$, ii) $d(x,y) = 0 \leftrightarrow x = y$, and ii) $d(x,y) \leq d(x,z) + d(y,z)$. The ordered pair (Z, d) is said to be a metric space. Assume X and Y are two non-empty subsets of a metric space (Z, d). The *Hausdorff distance* $d_H(X,Y)$ is given by:

$$\max\{\sup_{x \in X} \inf_{y \in Y} d(x,y), \sup_{y \in Y} \inf_{x \in X} d(x,y)\}.$$

It follows that the ordered pair $(\mathbb{IR}^+, d_H)$ is a metric space. Now, we introduce some concepts from system theory [17] and a modified notion of *quotient system* [10].

Definition 1 (System [17]): A system is a sextuple $(X, X_0, U, \longrightarrow, Y, H)$ consisting of:

- a set of states X;
- a set of initial states $X_0 \subseteq X$;
- a set of inputs U;
- a transition relation $\longrightarrow \subseteq X \times U \times X$;
- a set of outputs Y;
- an output map $H : X \to Y$.

When the output set Y of a system is endowed with a metric, it is called a metric system. A system is autonomous if the cardinality of its input set is at most one.

Definition 2 (Approximate Simulation Relation [17]): Consider two metric systems $S_a = (X_a, X_{a0}, U_a, \underset{a}{\longrightarrow}, Y_a, H_a)$ and $S_b = (X_b, X_{b0}, U_b, \underset{b}{\longrightarrow}, Y_b, H_b)$ with $Y_a = Y_b$, and let $\varepsilon \in \mathbb{R}_0^+$, where $\mathbb{R}_0^+$ represents the set of nonnegative real numbers. A relation $R \subseteq X_a \times X_b$ is an ε-approximate simulation relation from S_a to S_b if the following three conditions are satisfied:

1) $\forall x_{a0} \in X_{a0}, \exists x_{b0} \in X_{b0}$ such that $(x_{a0}, x_{b0}) \in R$;
2) $\forall (x_a, x_b) \in R,$ we have $d(H_a(x_a), H_b(x_b)) \leq \varepsilon$;
3) $\forall (x_a, x_b) \in R, (x_a, u_a, x'_a) \in \underset{a}{\longrightarrow}$ in S_a $\exists (x_b, u_b, x'_b) \in \underset{b}{\longrightarrow}$ in S_b satisfying $(x'_a, x'_b) \in R$.

We say that S_b ε-approximately simulates S_a, denoted by $S_a \preceq_{\mathcal{S}}^{\varepsilon} S_b$, if there exists an ε-approximate simulation relation R from S_a to S_b.

Definition 3 (Power Quotient System [10]): Let $S = (X, X_0, \varnothing, \longrightarrow, Y, H)$ be an autonomous system and R be an equivalence relation on X. The power quotient of S by R, denoted by $S_{/R}$, is the autonomous system $(X_{/R}, X_{/R,0}, \varnothing, \underset{/R}{\longrightarrow}, Y_{/R}, H_{/R})$ consisting of:

- $X_{/R} = X/R$;
- $X_{/R,0} = \{x_{/R} \in X_{/R} | x_{/R} \cap X_0 \neq \varnothing\}$;
- $(x_{/R}, u, x'_{/R}) \in \underset{/R}{\longrightarrow}$ if $\exists (x, u, x') \in \longrightarrow$ with $x \in x_{/R}$ and $x' \in x'_{/R}$;
- $Y_{/R} \subset 2^Y$;
- $H_{/R}(x_{/R}) = \bigcup_{x \in x_{/R}} H(x)$.

Lemma 1 ([10]): Let S be an autonomous metric system, R be an equivalence relation on X, and let the autonomous metric system $S_{/R}$ be the power quotient system of S by R. For any

$$\varepsilon \geq \max_{\substack{x \in x_{/R} \\ x_{/R} \in X/R}} d(H(x), H_{/R}(x_{/R})),$$

with d the Hausdorff distance over the set 2^Y, $S_{/R}$ ε-approximately simulates S, i.e. $S \preceq_{\mathcal{S}}^{\varepsilon} S_{/R}$.

Now, we modify Definition 3 and Lemma 1 for the case when an over approximation of $S_{/R}$ can be constructed, i.e. $\bar{S}_{/R}$.

Definition 4: (Approximate Power Quotient System [11]) Let $\mathcal{S} = (X, X_0, U, \longrightarrow, Y, H)$ be a system, R be an equivalence relation on X, and $\mathcal{S}_{/R} = (X_{/R}, X_{/R,0}, U_{/R}, \underset{/R}{\longrightarrow}, Y_{/R}, H_{/R})$ be the power quotient of $\mathcal{S}$ by R. An approximate power quotient of $\mathcal{S}$ by R, denoted by $\bar{\mathcal{S}}_{/R}$, is a system $(X_{/R}, X_{/R,0}, U_{/R}, \underset{/\bar{R}}{\longrightarrow}, \bar{Y}_{/R}, \bar{H}_{/R})$ such that, $\underset{/\bar{R}}{\longrightarrow} \supseteq \underset{/R}{\longrightarrow}$, $\bar{Y}_{/R} \supseteq Y_{/R}$, and $\bar{H}_{/R}(x_{/R}) \supseteq H_{/R}(x_{/R})$, $\forall x_{/R} \in X_{/R}$.

Corollary 1 ([11]): Let S be a metric system, R be an equivalence relation on X, and let the metric system $\bar{S}_{/R}$ be the approximate power quotient system of S by R. For any

$$\varepsilon \geq \max_{\substack{x \in x_{/R} \\ x_{/R} \in X_{/R}}} d(H(x), \bar{H}_{/R}(x_{/R})),$$

with d the Hausdorff distance over the set 2^Y, $\bar{S}_{/R}$ ε-approximately simulates S, i.e. $S \preceq_{\mathcal{S}}^{\varepsilon} \bar{S}_{/R}$.

C. Timed Safety Automaton

We present a formal definition for TSA. A TSA [12] is a directed graph extended with real-valued variables (called clocks) that model the logical clocks. We define C as a set of finitely many clocks. Clock constraints are used to restrict the behavior of the automaton. A clock constraint is a conjunctive formula of atomic constraints of the form $x \bowtie n$ or $x - y \bowtie n$ for $x, y \in C$, $\bowtie \in \{\leq, <, =, >, \geq\}$ and $n \in \mathbb{N}$. We use $\mathcal{B}(C)$ to denote the set of clock constraints.

Definition 5: (Timed Safety Automaton [12]) A timed safety automaton TSA is a sextuple $(L, \ell_0, \mathsf{Act}, C, E, \mathsf{Inv})$ where:

- L is a set of finitely many locations (or vertices);
- $\ell_0 \in L$ is the initial location;
- Act is the set of actions;
- C is a set of finitely many real-valued clocks;
- $E \subseteq L \times \mathcal{B}(C) \times \mathsf{Act} \times 2^C \times L$ is the set of edges;
- $\mathsf{Inv} : L \to \mathcal{B}(C)$ assigns invariants to locations.

The location invariants are restricted to constraints of the form: $c \leq n$ or $c < n$ where c is a clock and $n \in \mathbb{N}_{>0}$.

D. Problem Statement

Consider the system $S = (X, X_0, \varnothing, \longrightarrow, Y, H)$:

- $X = \mathbb{R}^n$;
- $X_0 = \mathbb{R}^n$;
- $(x, x') \in \longrightarrow$ iff $\xi_x(\tau(x)) = x'$ given by (1)-(2), and (10);
- $Y \subset \mathbb{R}^+$;
- $H : \mathbb{R}^n \to \mathbb{R}^+$ where $H(x) = \tau(x)$.

The output of the above system generates all possible sequences of inter-sample intervals of the concrete system (1)-(2) with the TM (10).

Problem 1: Provide a construction of power quotient systems $S_{/\mathcal{P}}$ of systems S as defined above.

Based on Definition 3, we propose to construct the system $S_{/\mathcal{P}} = (X_{/\mathcal{P}}, X_{/\mathcal{P},0}, \varnothing, \underset{/\mathcal{P}}{\longrightarrow}, Y_{/\mathcal{P}}, H_{/\mathcal{P}})$ where

- $X_{/\mathcal{P}} = \mathbb{R}^n_{/\mathcal{P}} := \{\mathcal{R}_1, \ldots, \mathcal{R}_q\}$;
- $X_{/\mathcal{P},0} = \mathbb{R}^n_{/\mathcal{P}}$;
- $(x_{/\mathcal{P}}, x'_{/\mathcal{P}}) \in \underset{/\mathcal{P}}{\longrightarrow}$ if $\exists x \in x_{/\mathcal{P}}, \exists x' \in x'_{/\mathcal{P}}$ such that $\xi_x(H(x)) = x'$ as determined by (1)-(2);
- $Y_{/\mathcal{P}} \subset 2^Y \subset \mathbb{IR}^+$, where $\mathbb{IR}^+$ represents the set of closed intervals $[a, b]$ such that $0 < a \leq b$;
- $H_{/\mathcal{P}}(x_{/\mathcal{P}}) = [\min_{x \in x_{/\mathcal{P}}} H(x), \max_{x \in x_{/\mathcal{P}}} H(x)] := [\underline{\tau}_{x_{/\mathcal{P}}}, \bar{\tau}_{x_{/\mathcal{P}}}]$.

The equivalence relation $\mathcal{P}$ on $\mathbb{R}^n$ partitions the state space of S (i.e. the ETC system) into the set $X_{/\mathcal{P}}$ with a finite cardinality. However, since the exact construction of $\mathcal{S}_{/\mathcal{P}}$ is in general impossible, we construct instead $\bar{\mathcal{S}}_{/\mathcal{P}}$ (see Definition 4). Later on, it will be shown that the constructed $\bar{S}_{/\mathcal{P}}$ is equivalent to a TSA.

III. Abstractions of event-triggered LTI systems

In this section, we introduce the framework to solve Problem 1 in the following order: 1) a proper definition of an equivalence relation $\mathcal{P}$ on $\mathbb{R}^n$, 2) a tractable approach to compute the output map $\bar{H}_{/\mathcal{P}}$ and its corresponding output set $\bar{Y}_{/\mathcal{P}}$, and 3) a reachability-based analysis to derive the discrete transitions among abstract states $x_{/\mathcal{P}}$.

A. State set

The construction of $X_{/\mathcal{P}}$ mainly relies on the following observation.

Remark 1: Consider that the right-hand side of (20) is used to analyze the sampling behavior of (15) instead of $\mathcal{F}_\omega(x,t)$. Then, the sampling periods of all states, located on a line that passes through the origin excluding the origin itself, are lower bounded by the same sampling period, i.e. $\tau'(x) = \tau'(\lambda x)$, $\forall \lambda \neq 0$.

This suggests a state space abstraction via partitioning it into a finite number of polyhedral cones (pointed at the origin) $\mathcal{R}_s$ where $s \in \{1,\dots,q\}$ and $\bigcup_{s=1}^{q} \mathcal{R}_s = \mathbb{R}^n$. This technique is proposed by [18], dividing each of the angular spherical coordinates of $x \in \mathbb{R}^n$: $\theta_1,\dots,\theta_{n-2} \in [0,\pi]$, $\theta_{n-1} \in [-\pi,\pi]$ into $\bar{m}$ (not necessarily equidistant) intervals resulting in $q = \bar{m}^{(n-1)}$ conic regions. Furthermore, since the term $x^T\Theta(t)x$ is quadratic in x, it suffices to analyze half of the state space (e.g. by taking $\theta_{n-1} \in [0,\pi]$). Hence, the equivalence relation $\mathcal{P}$ to construct the abstraction is given by: $(x,x') \in \mathcal{P} \Leftrightarrow \exists\, s \in \{1,\dots,q\}$ s.t. $x, x' \in \mathcal{R}_s$, where q is the number of equivalence classes. The equivalence classes of $\mathcal{P}$ are defined by polyhedral cones pointed at the origin given by $\mathcal{R}_s = \{x \in \mathbb{R}^2 |\ x^T Q_s x \geq 0\}$, $Q_s \in \mathcal{S}_2$ whenever $n = 2$ or $\mathcal{R}_s = \{x \in \mathbb{R}^n |\ E_s x \geq 0\}$, $E_s \in \mathcal{S}_{n\times p}$ otherwise.

B. Output Map

In this subsection, we present how to construct $\bar{H}_{/\mathcal{P}}$ and $\bar{Y}_{/\mathcal{P}}$. For all $x \in \mathcal{R}_s$, the output $\bar{Y}_{/\mathcal{P}} = \bar{H}_{/\mathcal{P}}(x)$ is equal to the time interval $[\underline{\tau}_s, \bar{\tau}_s]$ indicating $\tau(x) \in [\underline{\tau}_s, \bar{\tau}_s]$. We make use of the polytopic embedding technique proposed by [19]. In the space of real matrices, a sequence of convex polytopes is constructed around the matrix $\Phi(t)$. Doing so replaces the evaluation of (14) at infinitely many instants t by the evaluation of $\Phi_{\kappa,s}$ at finitely many vertices in the sequence of polytopes generated by $\Phi_{\kappa,s}$. Assume a scalar $\sigma > 0$ denoting a time instant for which the TM (10) is enabled in the whole state space, i.e. $\Phi(t) \succeq 0$. Consider $N_{\text{conv}} + 1$ is the number of vertices employed to define the polytope containing $\Phi(t)$ in a given time interval, and $l \geq 1$ denotes the number of time subdivisions considered in the time interval $[0,\sigma]$.

Lemma 2: Let $s \in \{1,\dots,q\}$. Consider a time instant $\underline{\tau}_s \in (0,\sigma]$, a scalar μ and a symmetric matrix Ψ satisfying (11). If $\underline{\Phi}_{(i,j),s} \preceq 0$ holds $\forall (i,j) \in \mathcal{K}_s = (\{0,\dots,N_{\text{conv}}\} \times \{0,\dots,\lfloor \frac{\underline{\tau}_s l}{\sigma} \rfloor\})$, then, it follows that $\Phi(t) \preceq 0$, $\forall t \in [0,\underline{\tau}_s]$ with Φ defined in (13) and

$$\underline{\Phi}_{(i,j),s} = \tilde{\underline{\Phi}}_{(i,j),s} + \eta I$$

$$\tilde{\underline{\Phi}}_{(i,j),s} = \begin{cases} \Sigma_{k=0}^{i} \hat{\underline{\Phi}}_{(i,j),s} (\frac{\sigma}{l})^k & , \quad j < \lfloor \frac{\underline{\tau}_s l}{\sigma} \rfloor \\ \Sigma_{k=0}^{i} \hat{\underline{\Phi}}_{(i,j),s} (\underline{\tau}_s - j\frac{\sigma}{l})^k & , \quad j = \lfloor \frac{\underline{\tau}_s l}{\sigma} \rfloor, \end{cases} \tag{22}$$

$$\begin{aligned} \hat{\underline{\Phi}}_{(0,j),s} &= \begin{bmatrix} \underline{L}_{0,j} & \check{\Pi}_j^T M^T \\ M\check{\Pi}_j & -\Psi \end{bmatrix}, \\ \hat{\underline{\Phi}}_{(k\geq 1,j),s} &= \begin{bmatrix} \underline{L}_{k,j} & \hat{\Pi}_j^T \frac{(A^{k-1})^T}{k!} M^T \\ M\frac{A^{k-1}}{k!}\hat{\Pi}_j & 0 \end{bmatrix}, \end{aligned} \tag{23}$$

and

$$\underline{L}_{0,j} = \check{\Pi}_j^T M \check{\Pi}_j - N + \tilde{L}_{0,j} \tag{24}$$

with

$$\tilde{L}_{0,j} = \begin{cases} W\mu \frac{\lambda_{\max}(E^TE)}{\lambda_{\max}^A} (j\frac{\sigma}{l})(e^{\lambda_{\max}^A j\frac{\sigma}{l}} - 1)I \\ \quad \text{for} \quad \lambda_{\max}^A \neq 0, \\ W\mu \frac{\lambda_{\max}(E^TE)}{\lambda_{\max}^A} (j\frac{\sigma}{l})^2 I \\ \quad \text{for} \quad \lambda_{\max}^A = 0, \end{cases} \tag{25}$$

$$\underline{L}_{1,j} = \check{\Pi}_j^T M \hat{\Pi}_j + \hat{\Pi}_j^T M \check{\Pi}_j + \tilde{L}_{1,j} \tag{26}$$

with

$$\tilde{L}_{1,j} = \begin{cases} W\mu \frac{\lambda_{\max}(E^TE)}{\lambda_{\max}^A} [(j\frac{\sigma}{l}) e^{\lambda_{\max}^A j\frac{\sigma}{l}} \lambda_{\max}^A \\ + e^{\lambda_{\max}^A j\frac{\sigma}{l}} - 1]I \quad \text{for} \quad \lambda_{\max}^A \neq 0, \\ W\mu(2j\frac{\sigma}{l})\lambda_{\max}(E^TE)I \quad \text{for} \quad \lambda_{\max}^A = 0, \end{cases} \tag{27}$$

$$\underline{L}_{2,j} = \check{\Pi}_j^T M \frac{A}{2!} \hat{\Pi}_j + \hat{\Pi}_j^T \frac{A^T}{2!} M \check{\Pi}_j + \hat{\Pi}_j^T M \hat{\Pi}_j + \tilde{L}_{2,j} \tag{28}$$

with

$$\tilde{L}_{2,j} = \begin{cases} W\mu \frac{\lambda_{\max}(E^TE)}{\lambda_{\max}^A} [(j\frac{\sigma}{l}) e^{\lambda_{\max}^A j\frac{\sigma}{l}} \frac{(\lambda_{\max}^A)^2}{2!} \\ + e^{\lambda_{\max}^A j\frac{\sigma}{l}} \lambda_{\max}^A]I \\ \quad \text{for} \quad \lambda_{\max}^A \neq 0, \\ W\mu\lambda_{\max}(E^TE)I \quad \text{for} \quad \lambda_{\max}^A = 0, \end{cases} \tag{29}$$

$$\begin{aligned} \underline{L}_{k\geq 3,j} \quad &= \check{\Pi}_j^T M \frac{A^{k-1}}{k!} \hat{\Pi}_j + \hat{\Pi}_j^T \frac{(A^{k-1})^T}{k!} M \check{\Pi}_j \\ &+ \hat{\Pi}_j^T (\Sigma_{i=1}^{k-1} \frac{(A^{i-1})^T}{i!} M \frac{A^{k-i-1}}{(k-i)!}) \hat{\Pi}_j + \tilde{L}_{k,j} \end{aligned} \tag{30}$$

with

$$\tilde{L}_{k\geq 3,j} = \begin{cases} W\mu \frac{\lambda_{\max}(E^TE)}{\lambda_{\max}^A} [(j\frac{\sigma}{l}) e^{\lambda_{\max}^A j\frac{\sigma}{l}} \frac{(\lambda_{\max}^A)^k}{k!} \\ + e^{\lambda_{\max}^A j\frac{\sigma}{l}} \frac{(\lambda_{\max}^A)^{k-1}}{(k-1)!}]I \\ \quad \text{for} \quad \lambda_{\max}^A \neq 0, \\ 0 \quad \text{for} \quad \lambda_{\max}^A = 0, \end{cases} \tag{31}$$

$$\eta \geq \max_{t' \in [0,\frac{\sigma}{l}], r \in \{0,\dots,l-1\}} \lambda_{\max}\left(\Phi(t' + r\frac{\sigma}{l}) - \Sigma_{k=0}^{N} \hat{\underline{\Phi}}_{k,r}(t')^k\right), \tag{32}$$

and

$$\tilde{\underline{\Phi}}_{(N_{\text{conv}},j)}(t') = \Sigma_{k=0}^{N_{\text{conv}}} \hat{\underline{\Phi}}_{k,j}(t')^k. \tag{33}$$

Then, using the S-procedure, the following theorem provides an approach to regionally reduce the conservatism involved in the $\underline{\tau}_s$ estimates obtained from Lemma 2.

Theorem 2 (Regional Lower Bound Approximation): Consider a scalar $\underline{\tau}_s \in (0,\sigma]$, a scalar μ and a symmetric matrix Ψ satisfying (11), and matrices $\underline{\Phi}_{\kappa,s}$, $\kappa = (i,j) \in \mathcal{K}_s$, defined as in Lemma 2. If there exist scalars $\underline{\alpha}_{\kappa,s} \geq 0$ (for $n = 2$) or symmetric matrices $\underline{U}_{\kappa,s}$ with nonnegative entries (for $n \geq 3$) such that for all $\kappa \in \mathcal{K}_s$ the following LMIs hold:

$$\begin{cases} \underline{\Phi}_{(i,j),s} + \begin{bmatrix} \underline{\alpha}_{(i,j),s} Q_s & 0 \\ 0 & 0 \end{bmatrix} \preceq 0 & \text{if } n = 2, \\ \underline{\Phi}_{(i,j),s} + \begin{bmatrix} E_s^T \underline{U}_{(i,j),s} E_s & 0 \\ 0 & 0 \end{bmatrix} \preceq 0 & \text{if } n \geq 3, \end{cases} \tag{34}$$

then, the inter-sample time (10) of the system (1)-(2) is regionally bounded from below by $\underline{\tau}_s$, $\forall x \in \mathcal{R}_s$.

A similar approach is followed to find the upper bounds $\bar{\tau}_s$, as outlined in Lemma 3 and Theorem 3 in the following.

Lemma 3: Let $s \in \{1, \ldots, q\}$. Consider a time instant $\bar{\tau}_s \in [\underline{\tau}_s, \sigma]$, a scalar μ and a matrix Ψ satisfying the LMI conditions given in Lemma 2. If $\bar{\Phi}_{(i,j),s} \preceq 0$ holds $\forall (i,j) \in \mathcal{K}_s = (\{0, \ldots, N_{\text{conv}}\} \times \{\lfloor \frac{\bar{\tau}_s l}{\sigma} \rfloor, \ldots, l-1\})$, then, it follows that $\Phi(t) \succeq 0$, $\forall t \in [\bar{\tau}_s, \sigma]$ with Φ defined in (13) and

$$\bar{\Phi}_{(i,j),s} = -\bar{\bar{\Phi}}_{(i,j),s} - \eta I,$$

$$\bar{\bar{\Phi}}_{(i,j),s} = \begin{cases} \sum_{k=0}^{i} L_{k,j} (\frac{(j+1)\sigma}{l} - \bar{\tau}_s)^k & \text{if } j = \lfloor \frac{\bar{\tau}_s l}{\sigma} \rfloor, \\ \sum_{k=0}^{i} L_{k,j} (\frac{\sigma}{l})^k & \text{if } j > \lfloor \frac{\bar{\tau}_s l}{\sigma} \rfloor, \end{cases}$$

where $L_{k,j}$ are given by (24)-(31) and η is defined in (32).

Theorem 3 (Regional Upper Bound Approximation): Consider a scalar $\bar{\tau}_s \in [\underline{\tau}_s, \sigma]$, a scalar μ and a symmetric matrix Ψ satisfying (11), and matrices $\bar{\Phi}_{\kappa,s}$, $\kappa = (i,j) \in \mathcal{K}_s$, defined as in Lemma 3. If there exist scalars $\bar{\alpha}_{\kappa,s} \geq 0$ (for $n = 2$) or symmetric matrices $\bar{U}_{\kappa,s}$ with nonnegative entries (for $n \geq 3$) such that for all $\kappa \in \mathcal{K}_s$ the following LMIs hold:

$$\begin{cases} \bar{\Phi}_{(i,j),s} - \begin{bmatrix} \bar{\alpha}_{(i,j),s} Q_s & 0 \\ 0 & 0 \end{bmatrix} \preceq 0 & \text{if } n = 2, \\ \bar{\Phi}_{(i,j),s} - \begin{bmatrix} E_s^T \bar{U}_{(i,j),s} E_s & 0 \\ 0 & 0 \end{bmatrix} \preceq 0 & \text{if } n \geq 3, \end{cases} \tag{35}$$

then, the inter-sample time (10) of the system (1)-(2) is regionally bounded from above by $\bar{\tau}_s$, $\forall x \in \mathcal{R}_s$.

C. Transition Relations

To find the transitions, it is required to compute the reachable set of each $\mathcal{R}_s$ over $[\underline{\tau}_s, \bar{\tau}_s]$. Using the Minkowski sum, we present a way to compute over approximations of the reachable sets. Denote by $\xi_x(\tau) = \Lambda(\tau)x + \Omega(\tau)$ and $\mathcal{X}_{[\underline{\tau}_s, \bar{\tau}_s]}(X_{0,s})$ the state evolution and the reachable set of $X_{0,s}$ over $[\underline{\tau}_s, \bar{\tau}_s]$, respectively, where $\mathcal{X}_{[\underline{\tau}_s, \bar{\tau}_s]}(X_{0,s}) = \{x' \in \mathbb{R}^n \mid \exists x \in X_{0,s}, \exists \tau \in [\underline{\tau}_s, \bar{\tau}_s], x' = \xi_x(\tau)\}$. Define:

$$\begin{aligned} &\mathcal{X}^1_{[\underline{\tau}_s, \bar{\tau}_s]}(X_{0,s}) \\ &:= \{x' \in \mathbb{R}^n \mid \exists x \in X_{0,s}, \exists \tau \in [\underline{\tau}_s, \bar{\tau}_s], x' = \Lambda(\tau)x\}, \\ &\mathcal{X}^2_{[\underline{\tau}_s, \bar{\tau}_s]}(X_{0,s}) \\ &:= \{x' \in \mathbb{R}^n \mid \exists x \in X_{0,s}, \exists \tau \in [\underline{\tau}_s, \bar{\tau}_s], x' = \Omega(\tau)\}. \end{aligned}$$

It follows that:

$$\mathcal{X}_{[\underline{\tau}_s, \bar{\tau}_s]}(X_{0,s}) := \mathcal{X}^1_{[\underline{\tau}_s, \bar{\tau}_s]}(X_{0,s}) \bigoplus \mathcal{X}^2_{[\underline{\tau}_s, \bar{\tau}_s]}(X_{0,s}).$$

In [10, Section III.B.3], it has been shown that it is enough to consider subsets $X_{0,s} \subset \mathcal{R}_s$ being convex polytopes with each vertex placed on each of the extreme rays of $\mathcal{R}_s$ (excluding the origin) to compute $\mathcal{X}^1_{[\underline{\tau}_s, \bar{\tau}_s]}$. Then, one can compute an over approximation of $\mathcal{X}^1_{[\underline{\tau}_s, \bar{\tau}_s]}$, denoted by $\hat{\mathcal{X}}_{[\underline{\tau}_s, \bar{\tau}_s]}$. Furthermore, it follows that:

$$\begin{aligned} \|\Omega(\tau)\| &= \|\textstyle\int_0^\tau e^{A(\tau-s)} E\omega(s) ds\| \\ &\leq \textstyle\int_0^\tau \|e^{A(\tau-s)} E\omega(s)\| ds \\ &\leq \textstyle\int_0^\tau \|e^{A(\tau-s)}\| \|E\| |\omega(s)| ds \\ &\leq W|x| \|E\| \textstyle\int_0^\tau |e^{\mu(A)(\tau-s)}| ds \quad = \rho(\tau)|x| \end{aligned}$$

where $\rho(\tau) = W\|E\| \int_0^\tau |e^{\mu(A)(\tau-s)}| ds$. Thus, it follows that $\mathcal{X}^2_{[\underline{\tau}_s, \bar{\tau}_s]}$ can be over approximated by $\hat{\mathcal{X}}^2_{[\underline{\tau}_s, \bar{\tau}_s]}(X_{0,s})$ given by: $\{x' \in \mathbb{R}^n \mid \exists x \in X_{0,s}, \exists \tau \in [\underline{\tau}_s, \bar{\tau}_s], |x'| \leq \rho(\bar{\tau}_s)|x|\}$.

To compute the transitions in $\bar{S}_{/\mathcal{P}}$, it thus suffices to derive the intersection between the over approximation $\hat{\mathcal{X}}_{[\underline{\tau}_s, \bar{\tau}_s]}(X_{0,s})$ and all the conic regions $\mathcal{R}_t$ where $t \in \{1, \ldots, q\}$. To do so, it is required to check if the following convex feasibility problem for each conic region $\mathcal{R}_t$ holds:

$$\mathcal{R}_t \cap \hat{\mathcal{X}}_{[\underline{\tau}_s, \bar{\tau}_s]}(X_{0,s}) \neq \varnothing. \tag{36}$$

Hence, there exists a transition from abstract state $\mathcal{R}_s$ to $\mathcal{R}_t$ in $\bar{S}_{/\mathcal{P}}$ in the case that (36) is satisfied.

D. Timed Safety Automata Representation

First, we point out the connection between an abstract state $x_{/\mathcal{P}} \in X_{/\mathcal{P}}$ and its corresponding output $y_{/\mathcal{P}} \in \bar{Y}_{/\mathcal{P}}$ [10]. The system $\bar{S}_{/\mathcal{P}}$ remains at $x_{/\mathcal{P}}$ during $[0, \underline{\tau}_{x_{/\mathcal{P}}})$, possibly leaves $x_{/\mathcal{P}}$ during $[\underline{\tau}_{x_{/\mathcal{P}}}, \bar{\tau}_{x_{/\mathcal{P}}})$, and is forced to leave $x_{/\mathcal{P}}$ at $\bar{\tau}_{x_{/\mathcal{P}}}$. Thus, the semantics of $\bar{S}_{/\mathcal{P}}$ is equivalent to a TSA given by $\mathsf{TSA} = (L, \ell_0, \mathsf{Act}, C, E, \mathsf{Inv})$ where:

- $L = X_{/\bar{\mathcal{P}}}$;
- $\ell_0 := \mathcal{R}_s$ such that $\xi(0) \in \mathcal{R}_s$;
- $\mathsf{Act} = \{*\}$ is an arbitrary symbol;
- $C = \{c\}$;
- E is given by all tuples $(\mathcal{R}_s, g, a, r, \mathcal{R}_t)$ such that $(\mathcal{R}_s, \mathcal{R}_t) \in \xrightarrow[/\bar{\mathcal{P}}]{}$, $g = \{c \mid c \in [\underline{\tau}_s, \bar{\tau}_s]\}$, $a = *$, and r is given by $c := 0$;
- $\mathsf{Inv}(\mathcal{R}_s) := \{c \mid c \in [0, \bar{\tau}_s]\}, \forall s \in \{1, \ldots, q\}$.

On the complexity of our approach, although the construction technique presented in this section is offline, it is exponentially dependent on $n-1$ (where n is the number of states).

IV. Numerical Example

We illustrate the theoretical results of this paper in a numerical example. Consider an LTI sytem, used as an example in [1], and add a perturbation term $\omega(t)$ as follows:

$$\dot{\xi}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \xi(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \nu(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \omega(t) \tag{37}$$

with $W = 0.001$, $\gamma = 100$, and $\beta = 0.25$, see (3), (7)-(8) and (10)-(9), respectively. Then, solving the ARE associated with the $\mathcal{L}_2$ stability, the control update law is computed:

$$\nu(t) = -K\xi(t_k) = -[0.2361 \quad 6.2367]\xi(t_k), \forall t \in [t_k, t_{k+1})$$

where t_k denotes the sampling instants and $k \in \mathbb{N}_0$. We set the order of polynomial approximation $N_{\text{conv}} = 7$, the number of polytopic subdivisions $l = 800$, the upper bound of the inter-sample intervals $\sigma = 8$, the number of angular sub-divisions $\bar{m} = 10$, thus, $q = 2 \times 10^{(2-1)} = 20$. Then, applying the results from Section III-B, we get the precision abstraction of $\varepsilon = 6.100$. Compared to the results found in [11], the derived ε is large. However, one must take into account the possible stabilizing effect of a disturbance on the dynamics in (1) enlarges the derived $\bar{\tau}_s$ and a more thorough study is due in this regard. In Figure 1, the derived lower and upper bounds are depicted. It is evident that the derived $\underline{\tau}_s$ compared to the MIET are less conservative and can be

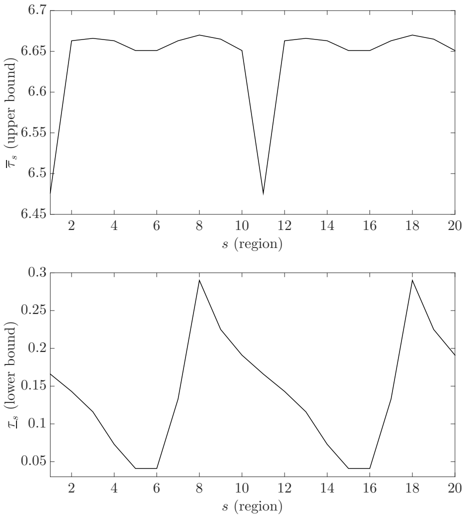

Fig. 1. (Top plot) Upper bounds on regional inter-sample times. (Bottom plot) Lower bounds on regional inter-sample times.

effectively used for scheduling. Figure 2 represents the conic regions s and the associated $\underline{\tau}_s$ and $\bar{\tau}_s$ (note that in order to show the lower bounds in a clear manner the lower and upper bounds are depicted, separately). Figure 3 depicts the result of applying the procedure introduced in Section III-C. It is clear that due to the large ε some discrete states have many out-going transitions.

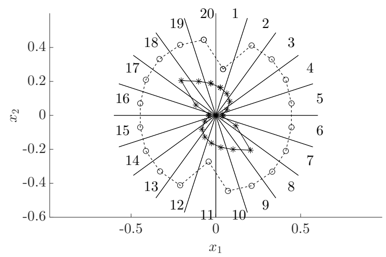

Fig. 2. The radial distance from the origin of each asterisk indicates the regional lower bound of the indexed cone. Furthermore, in the case of circles, the distance indicates the regional upper bound of the indexed cone minus 6.2 sec, i.e. $\bar{\tau}_s - 6.2\,\text{sec}$ (for the sake of clarity of the figure).

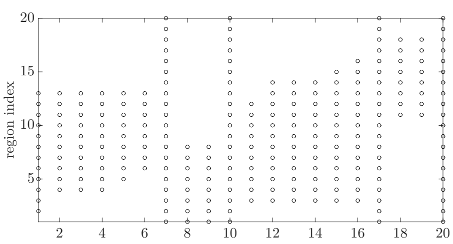

Fig. 3. Schematic representation of set of edges in the timed automaton generated by (37)-(IV). A circle at the coordinate (i, j) denotes an edge from location i to location j.

V. Conclusions

We have presented an approach to capture the sampling behavior of perturbed LTI systems with an $\mathcal{L}_2$-based TM by a TSA. The derived TSA ε-approximately simulates the ETC system and can be used to synthesize schedulers for ETC feedback loops. In simple words, instead of considering the MIET of ETC system as an indicator of the maximum bandwidth utilization, we have proposed a way to construct a TSA that enjoys much less conservative quantities for scheduling of ETC feedback loops. Employing synthesis tools of TSA's, one can further extend our results to synthesize conflict-free policies in WNCS's, see e.g. [10]. However, in [10], a centralized scheduler is proposed, hence, another promising direction is to search for a decentralized scheduler rather than a centralized one.

VI. Acknowledgments

The authors gratefully appreciate the fruitful discussions with Dieky Adzkiya and Nikolaos Kekatos.

References

[1] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Trans. Autom. Control*, vol. 52, no. 9, pp. 1680–1685, Sep. 2007.

[2] M. Velasco, J. Fuertes, and P. Marti, "The self triggered task model for real-time control systems," in *Work-in-Progress Session of the 24th IEEE Real-Time Systems Symposium*, vol. 384, 2003.

[3] E. W. Dijkstra, "On the role of scientific thought," in *Selected writings on computing: a personal perspective*. Springer, 1982, pp. 60–66.

[4] G. Buttazzo, G. Lipari, and L. Abeni, "Elastic task model for adaptive rate control," in *Proc. 19th IEEE Real-Time Systems Symposium*, Dec. 1998, pp. 286–295.

[5] C. Lu, J. Stankovic, S. Son, and G. Tao, "Feedback control real-time scheduling: Framework, modeling, and algorithms," *Real-Time Systems*, vol. 23, no. 1-2, pp. 85–126, 2002.

[6] R. Bhattacharya and G. Balas, "Anytime control algorithm: Model reduction approach," *Journal of Guidance, Control, and Dynamics*, vol. 27, no. 5, pp. 767–776, 2004.

[7] D. Fontanelli, L. Greco, and A. Bicchi, "Anytime control algorithms for embedded real-time systems," in *Hybrid Systems: Computation and Control*, 2008, pp. 158–171.

[8] S. Al-Areqi, D. Gorges, S. Reimann, and S. Liu, "Event-based control and scheduling codesign of networked embedded control systems," in *Proc. Amer. Control Conf.*, Jun. 2013, pp. 5299–5304.

[9] S. Al-Areqi, D. Gorges, and S. Liu, "Stochastic event-based control and scheduling of large-scale networked control systems," in *Proc. European Control Conf.*, Jun. 2014, pp. 2316–2321.

[10] A. Sharifi Kolarijani, D. Adzkiya, and M. Mazo Jr., "Symbolic abstractions for the scheduling of event-triggered control systems," in *Proc. 54th IEEE Conference on Decision and Control*, Dec 2015, pp. 6153–6158.

[11] A. Sharifi Kolarijani and M. Mazo Jr, "A formal traffic characterization of LTI event-triggered control systems," *to appear in IEEE Trans. Control of Network Systems, arXiv:1503.05816*, 2015.

[12] T. A. Henzinger, X. Nicollin, J. Sifakis, and S. Yovine, "Symbolic model checking for real-time systems," *Information and computation*, vol. 111, no. 2, pp. 193–244, 1994.

[13] R. Alur and D. Dill, "A theory of timed automata," *Theoretical Computer Science*, vol. 126, no. 2, pp. 183–235, 1994.

[14] K. G. Larsen, P. Pettersson, and W. Yi, "Uppaal in a nutshell," *International Journal on Software Tools for Technology Transfer*, vol. 1, no. 1, pp. 134–152, 1997.

[15] X. Wang and M. D. Lemmon, "Self-triggered feedback control systems with finite-gain $\mathcal{L}_2$ stability," *IEEE Trans. Autom. Control*, vol. 54, no. 3, pp. 452–467, 2009.

[16] A. Sharifi Kolarijani, M. Mazo Jr., and T. Keviczky, "Technical report: Timing abstractions of perturbed LTI systems with $\mathcal{L}_2$-based event-triggering mechanism," *CoRR*, vol. abs/1609.03476, 2016.

[17] P. Tabuada, *Verification and Control of Hybrid Systems: A Symbolic Approach*. Springer London, Limited, 2009.

[18] C. Fiter, L. Hetel, W. Perruquetti, and J.-P. Richard, "A state dependent sampling for linear state feedback," *Automatica*, vol. 48, no. 8, pp. 1860–1867, 2012.

[19] L. Hetel, J. Daafouz, and C. Iung, "Stabilization of arbitrary switched linear systems with unknown time-varying delays," *Automatic Control, IEEE Transactions on*, vol. 51, no. 10, pp. 1668–1674, 2006.